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Spectral Analysis Of Self-Adjoint Second Order Differential Operators

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Abstract

The primary purpose of this study is to investigate the asymptotic distribution of the eigenvalues of self-adjoint second order differential operators. We study the following differential equation:

$$-y'' + gy' + hy = \lambda y$$

on the interval $[0, a]$ where $a > 0$ and $g, h \in C^1[0, a]$. We first analyse the problem where the functions g and h are equal to zero. To improve on the terms of the eigenvalue problem for $g, h = 0$, we consider the eigenvalue problem for general functions g and h. Here we calculate explicitly the first four terms of the eigenvalue asymptotics problem.

Declaration

I, Norman Boshego declare that the contents of this dissertation are original except where due references are made. It is submitted for the degree of Master of Science at the University of the Witwatersrand, Johannesburg, South Africa. It has not been submitted before for any degree or examination to any other institution.

Norman Boshego
March, 2015

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Contents

Chapter 1

INTRODUCTION AND RELATED WORK

1.1 Background introduction

Differential equations were initially used to describe fundamental laws in Physics and are currently used in areas such as Engineering, Economics, Geography and other areas including Financial engineering. In Financial engineering differential equations are widely used for pricing derivatives. In modern days differential equations are also used to study electricity, mechanics, climate changes, economic trends and population forecasting.

There is an application of differential operators in the study of differential equations. Differential operators are of extensive scope which covers several topics. The topics covered among others are eigenvalues, boundary conditions, asymptotics, adjoint operators and self-adjoint operators.

1.2 Literature review

In this section we look at papers that are related to the study to be undertaken.

1.

Shibata in [16] considered a two parameter nonlinear Sturm-Liouville equation:

$$u''(x) + \mu u(x)^p = \lambda u(x)^q, \quad x \in (0,1), \tag{1.2.1}$$

$$u(x) > 0, \quad x \in (0,1),$$

$$u(0) = u(1) = 0,$$

where $1 < q < p < q + 2$ and $\mu, \lambda > 0$ are eigenvalue parameters. The objective of the paper was to investigate asymptotic properties of eigenvalues. This problem is not necessarily related to the one under study because of its non-linearity.

2.

Birman and Solomyak considered in [6] a lower semibounded boundary-value problem of the form

$$A(x, D)u = \lambda u, \tag{1.2.2}$$

where $A(x, D)$ is a self-adjoint elliptic operator. Denote by $N(\lambda)$ the number of eigenvalues of the problem in the interval $(-\infty, \lambda)$. A general hypothesis regarding the form of $N(\lambda)$ consists in the following. Let the real-valued function $\hat{a}(x, \xi)$ be the symbol of the operator A in the domain $\Omega \subset \mathbb{R}^m$.
Then as $\lambda \to \infty$

$$N(\lambda) \sim (2\pi)^{-m} mes\{(x, \xi) \in \Omega \times \mathbb{R}^m : \hat{a}(x, \xi) < \lambda\}, \tag{1.2.3}$$

where mes denotes a Lebesgue measure.
The more general form

$$B(x, D)u = \mu A(x, D)u \tag{1.2.4}$$

of (1.2.2) used in many cases was also considered.
In equation (1.2.4), the operator A is always considered to be of higher or-

der than B. There is no assumption of the operator B to be of definite sign. Hence, generally, equation (1.2.4) has eigenvalues of both signs. Denote by $\tilde{N}_+(\mu), \tilde{N}_-(\mu), \mu > 0$, the number of eigenvalues of equation (1.2.4) in the interval $(\mu, \infty), (-\infty, \mu)$.
The equation (1.2.2) generalizes to the case of problems of form (1.2.4). The change from λ to μ^{-1} should not be forgotten and this corresponds to the asymptotic behavior as $\mu \to 0$.

Also considered in their paper is an asymptotic estimate for the eigenvalue counting function for semibounded elliptic operators of higher order. A method for finding uniform estimates for the parabolic's Green function was developed. The method developed made it easy and possible, particularly to consider self-adjoint equations of form (1.2.2) in $\mathbb{R}^m$, where

$$A(x, D) = L_0(x, D) + L_1(x, D) + q(x), \tag{1.2.5}$$

where L_0 is a homogeneous, uniformly elliptic operator of order l with bounded Lipshitz continuous coefficients; the function $q(x) \to +\infty$ as $|x| \to \infty$ and satisfies a few conditions common for the Tauberian technique. The operator $L_1(x, D)$ of order less than l is subordinate to the operator L_0+q. The following estimate is obtained:

$$(2\pi)^m \Gamma(ml^{-1} + 1) N(\lambda) \sim \int_{\mathbb{R}^m \times \mathbb{R}^m} [\lambda - q(x)]_+^{m/l} exp[-\hat{L}_0(x, \xi)] d\xi dx. \tag{1.2.6}$$

$\hat{L}_0$ is the symbol of the operator L_0.
In their paper, Birman and Solomyak also considered a natural generalization of the Schrödinger operator,

$$My = -y'' + Q(x)y, \tag{1.2.7}$$

considered in the space of vector-valued functions $L_2(\mathbb{R}, H)$, where H is a separable Hilbert space, $Q(x)$ is an operator-valued function with behavior which determines the character of the spectrum operator (1.2.7). Particularly, if for each x, the operator $Q(x)$ is self-adjoint and has discrete spectrum$\{\lambda_n(x)\}$, while $\lambda_1(x) \to +\infty$ as $|x| \to \infty$, then the spectrum of M is also discrete.

Under particular restrictions on $Q(x)$ it was shown that

$$N(\lambda) \sim \pi^{-1} \sum_n \int [\lambda - \lambda_n(x)]_+^{1/2} dx. \tag{1.2.8}$$

Equation (1.2.8) may be considered as the translation of

$$\tilde{N}_\pm(\mu) \sim (2\pi)^{-m} \int_{\Omega \times \mathbb{R}^m} \tilde{n}_\pm(x, \xi; \mu) d\xi dx, \tag{1.2.9}$$

where $\Omega \subset \mathbb{R}^m$ and $\tilde{n}_\pm(x, \xi; \mu)$ are the distribution functions of the positive and negative eigenvalues of the algebraic problem

$$\mu \hat{a}(x, \xi) f = \hat{b}(x, \xi) f, \quad f \in C^k, \tag{1.2.10}$$

and the symbols $\hat{a}, \hat{b}$ are Hermitian $k \times k$ matrices. We consequently have equation (1.2.9) from equation (1.2.10) as $\mu \to +0$.

3.

Here we look at a paper by Faierman, [8]. He considered two simultaneous Sturm-Liouville systems, one defined on the interval $0 \leq x_1 \leq 1$, the second on the interval $0 \leq x_2 \leq 1$, and each containing the parameters λ and μ. The eigenvalues and eigenfunctions of the simultaneous system are denoted by $(\lambda_{j,k}, \mu_{j,k})$ and $\psi_{j,k}(x_1, x_2)$, $j, k = 0, 1, \ldots$. Asymptotic methods are used to derive asymptotic formulae for these expressions as $j + k \to \infty$, when (j, k) strictly lie in a particular portion of (x, y)-plane.

The usefulness of multiparameter Sturm-Liouville problems in mathematical physics has led to a revival of interest in this area of study. Faierman investigated the behavior of the eigenvalues and eigenfunctions of the simultaneous two-parameter systems

$$y_1'' + (\lambda A_1(x_1) - \mu B_1(x_1) + q_1(x_1))y_1 = 0, \quad x_1 \in [0,1], \tag{1.2.11}$$

$$\begin{aligned} y_1(0)\cos\alpha_1 - y_1'(0)\sin\alpha_1 = 0, &\quad \alpha_1 \in [0,\pi), \\ y_1(1)\cos\beta_1 - y_1'(1)\sin\beta_1 = 0, &\quad \beta_1 \in (0,\pi], \end{aligned} \tag{1.2.12}$$

and

$$y_2'' + (\lambda A_2(x_2) - \mu B_2(x_2) + q_2(x_2))y_2 = 0, \quad x_2 \in [0,1], \tag{1.2.13}$$

$$\begin{aligned} y_2(0)\cos\alpha_2 - y_2'(0)\sin\alpha_2 = 0, &\quad \alpha_2 \in [0,\pi), \\ y_2(1)\cos\beta_2 - y_2'(1)\sin\beta_2 = 0, &\quad \beta_2 \in (0,\pi]. \end{aligned} \tag{1.2.14}$$

Faierman stated that by an eigenvalue of the system $(1.2.11 - 1.2.14)$, means a pair of numbers, (λ^*, μ^*), such that when $\lambda = \lambda^*$ and $\mu = \mu^*$, (1.2.11) and (1.2.13) have nontrivial solutions; say $y_i(x_i, \lambda^*, \mu^*)$ which satisfy (1.2.12) and (1.2.14).
Furthermore, the eigenvalues and normalized eigenfunctions of the system $(1.2.11)-(1.2.14)$ may be represented in the form $(\lambda_{j,k}, \mu_{j,k})$ and $\psi_{j,k}(x_1, x_2)$ respectively, $j, k = 0, 1, ...,$ where $(\lambda_{j,k}, \mu_{j,k})$ means that eigenvalue of $(1.2.11) - (1.2.14)$ for which $y_1(x_1, \lambda_{j,k}, \mu_{j,k})$ has precisely j zeros in $0 < x_1 < 1$ and $y_2(x_2, \lambda_{j,k}, \mu_{j,k})$ has precisely k zeros in $0 < x_2 < 1$, while

$$\psi_{j,k}(x_1, x_2) = C_{j,k} \prod_{i=1}^{2} y_i(x_i, \lambda_{j,k}, \mu_{j,k})$$

and $C_{j,k}$ denotes a normalization constant.

4.

In [17], Shkalikov considered the boundary problem

$$l(y,\lambda) = y^{(n)} + p_1(x,\lambda)y^{(n-1)} + ... + p_n(x,\lambda)y = 0, \tag{1.2.15}$$

$$(U_j(y,\lambda)) = \sum_{k=0}^{n-1} a_{jk}(\lambda)y^{(k)}(0) + b_{jk}(\lambda)y^{(k)}(1) = 0, \quad j = 1,...,n, \tag{1.2.16}$$

in the interval $[0,1]$, where $p_s(x,\lambda) = \sum_{\nu=0}^{s} p_{\nu s}(x)\lambda^{\nu}$, $p_{ss} = constant$, $s = 1,...,n$, $p_{nm} \neq 0$, and $a_{jk}(\lambda), b_{jk}(\lambda)$ are arbitrary polynomials in λ.

In general, the spectral properties of (1.2.15) and (1.2.16) are mainly determined not only by the boundary conditions, but also by the highest coefficients of the polynomial in λ, $p_s(x,\lambda)$, $s = 1,...,n$. Hence for the same boundary conditions, but different functions $p_s(x,\lambda)$, the problems can be both regular and non-regular.

5.

Lastly and most importantly, we look at a paper by Möller and Zinsou. This is the paper that corresponds mostly to the work carried out in this research project. Möller and Zinsou, in [13], considered a fourth-order regular ordinary differential operator on the interval $[0,a]$ with eigenvalue dependent boundary conditions. The eigenvalue problem considered is,

$$y^{(4)} - (gy')' = \lambda^2 y, \tag{1.2.17}$$

$$B_j(\lambda)y = 0, \quad j = 1,2,3,4, \tag{1.2.18}$$

where $a > 0$, $g \in C^1[0,a]$ is a real-valued function and (1.2.18) are separated boundary conditions where the $B_j(\lambda)$ are constant or depend on λ linearly. The quasi-derivatives associated with (1.2.18) are given by

$$y^{[0]} = y, \quad y^{[1]} = y', \quad y^{[2]} = y'', \quad y^{[3]} = y^{(3)} - gy', \quad y^{[4]} = y^{(4)} - (gy')'.$$

The boundary conditions (1.2.18) are taken at the endpoint 0 for $j = 1,2$ and at the endpoint a for $j = 3, 4$. Furthermore, it was assumed for simplicity that either $B_j(\lambda)y = y^{[p_j]}(a_j) + i\epsilon_j \alpha\lambda y^{[q_j]}(a_j)$, or $B_j(\lambda)y = y^{[p_j]}(a_j)$, where $a_j = 0$ for $j = 1, 2$, $a_j = a$ for $j = 3, 4$, $\alpha > 0$ and $\epsilon_j \in \{-1, 1\}$. They defined

$$\Theta_1 = \{s \in 1,2,3,4 : B_s(\lambda) \quad \text{depends on} \quad \lambda\}, \quad \Theta_0 = \{1,2,3,4\} \setminus \Theta_1,$$
$$\Theta_1^0 = \Theta_1 \bigcap \{1,2\}, \quad \Theta_1^a = \Theta_1 \bigcap \{3,4\}$$

and assumed that the numbers p_1, p_2, q_j for $j \in \Theta_1^0$ and p_3, p_4, q_j for $j \in \Theta_1^a$ are distinct.

Möller and Zinsou firstly considered eigenvalue asymptotics for $g = 0$, and developed a formula for the asymptotic distribution of the eigenvalues, which is used to obtain the corresponding formula for general g.

Let

$$M(\lambda) = (B_i(\lambda) y_j(., \lambda))_{i,j=1}^4, \tag{1.2.19}$$

where $\lambda = \mu^2$ and

$$y(x, \mu) = \frac{1}{2\mu^3} \sinh(\mu x) - \frac{1}{2\mu^3} \sin(\mu x)$$

we denote

$$y_j(x, \lambda) = y^{(4-j)}(x, \mu), \quad j = 1, 2, 3, 4.$$

The first and second rows of $M(\lambda)$ have exactly one entry 1 and all other entries are zero. It follows that $\det M(\lambda) = \pm\phi(\mu)$, where

$$\phi(\mu) = \det \begin{pmatrix} B_3(\mu^2) y^{(r_1)}(., \mu) & B_3(\mu^2) y^{(r_2)}(., \mu) \\ B_4(\mu^2) y^{(r_1)}(., \mu) & B_4(\mu^2) y^{(r_2)}(., \mu) \end{pmatrix} \tag{1.2.20}$$

where $r_1 < r_2$ and $\{0, 1, 2, 3\} \setminus \{p_1, p_2\} = \{3 - r_2, 3 - r_1\}$. In view of $p_1 + p_2 \neq 3$ it follows that $r_1 = p_1$ and $r_2 = p_2$.

Therefore

$$\begin{aligned}\phi(\mu) &= y^{(p_1+2)}(a,\mu)y^{(p_2+3)}(a,\mu) - y^{(p_1+3)}(a,\mu)y^{(p_2+2)}(a,\mu)\\ &+ i\alpha\mu^2[-y^{(p_1+2)}(a,\mu)y^{(p_2)}(a,\mu) + y^{(p_1+1)}(a,\mu)y^{(p_2+3)}(a,\mu)\\ &+ y^{(p_1)}(a,\mu)y^{(p_2+2)}(a,\mu) - y^{(p_1+3)}(a,\mu)y^{(p_2+1)}(a,\mu)]\\ &+ \alpha^2\mu^4[y^{(p_1+1)}(a,\mu)y^{(p_2)}(a,\mu) - y^{(p_1)}(a,\mu)y^{(p_2+1)}(a,\mu)].\end{aligned} \tag{1.2.21}$$

The highest μ-power of ϕ occurs with

$$i\alpha\mu^2[y^{(p_1+1)}(a,\mu)y^{(p_2+3)} - y^{(p_1+3)}(a,\mu)y^{(p_2+1)}].$$

Hence, they first investigated the zeros of

$$\phi_0(\mu) = 2\mu^2[y^{(p_1+1)}(a,\mu)y^{(p_2+3)} - y^{(p_1+3)}(a,\mu)y^{(p_2+1)}], \tag{1.2.22}$$

was done. For the four possible cases, the following were obtained.
Case 1: $p_1 = 0$, $p_2 = 1$:

$$\phi_0(\mu) = -\mu[\cosh(\mu a)\sin(\mu a) + \sinh(\mu a)\cos(\mu a)].$$

Case 2: $p_1 = 0$, $p_2 = 2$:

$$\phi_0(\mu) = -2\mu^2[\cosh(\mu a)\cos(\mu a)].$$

Case 3: $p_1 = 1$, $p_2 = 3$:

$$\phi_0(\mu) = 2\mu^4[\sinh(\mu a)\sin(\mu a)].$$

Case 4: $p_1 = 2$, $p_2 = 3$:

$$\phi_0(\mu) = \mu^5[\cosh(\mu a)\sin(\mu a) + \sinh(\mu a)\cos(\mu a)].$$

We only consider case 1. Then we have that ϕ_0 has a zero of multiplicity 2 at 0, exactly one simple zero $\tilde{\mu_k}$ in each interval $((k-\frac{1}{2})\frac{\pi}{a},(k+\frac{1}{2})\frac{\pi}{a})$ for positive integers k with asymptotics

$$\tilde{\mu_k} = (4k-1)\frac{\pi}{4a} + o(1), \quad k = 1, 2, ...$$

simple zeros at $-\tilde{\mu}_k$, $\tilde{\mu}_{-k} = i\tilde{\mu}_k$ and $-i\tilde{\mu}_k$ for $k = 1, 2, ...$, and no other zeros.

The proof of case 1 will be complete if we show that all zeros of ϕ_0 lie on the real or imaginary axis. The product-to-sum formula for trigonometric functions gives

$$\begin{aligned}\phi_0 &= -\mu[\cosh(\mu a)\sin(\mu a) + \sinh(\mu a)\cos(\mu a)]\\ &= -\frac{1}{2}\mu[\sin((1+i)\mu a) + \sin((1-i)\mu a) - i\sin((1+i)\mu a) + i\sin((1-i)\mu a)]\\ &= -\frac{1}{2}\mu[(1-i)\sin((1+i)\mu a) + (1+i)\sin((1-i)\mu a)]. \end{aligned} \tag{1.2.23}$$

Putting $(1+i)\mu a = x+iy$, $x, y \in \mathbb{R}$, it follows for $\mu \neq 0$ that

$$\begin{aligned}\phi_0(\mu) = 0 &\Rightarrow |\sin((1+i)\mu a)| = |\sin((1-i)\mu a)|\\ &\Leftrightarrow |\sin(x+iy)| = |\sin(y-ix)|\\ &\Leftrightarrow \cosh^2 y - \cos^2 x = \cosh^2 x - \cos^2 y\\ &\Leftrightarrow \cosh^2(|y|) - \cos^2(|y|) = \cosh^2(|x|) - \cos^2(|x|). \end{aligned} \tag{1.2.24}$$

Since $\cosh^2 x + \cos^2 x = \frac{1}{2}\cosh(2x) + \frac{1}{2}\cos(2x) + 1$ has positive derivative on $(0,\infty)$, this function is strictly increasing, and $\phi_0(\mu)$ therefore implies by (1.2.24) that $|y| = |x|$ and thus $y = \pm x$. Then

$$\mu = \frac{x+iy}{(1+i)a} = \frac{(1 \pm i)}{(1+i)}\frac{x}{a}$$

is either real or pure imaginary.

Möller and Zinsou improved the asymptotics of the eigenvalues for when g in (1.2.17) is a general function. They established a more precise asymptotics for the eigenvalues. Again we set $\lambda = \mu^2$. They showed that equation (1.2.17) has an asymptotic fundamental system $\{\eta_1, \eta_2, \eta_3, \eta_4\}$ of the form

$$\eta_\nu^{(j)}(x,\mu) = \delta_{\nu,j}(x,\mu)\exp(i^{\nu-1\mu x}) \tag{1.2.25}$$

where $\delta_{\nu,j}$ have the expansion

$$\delta_{\nu,j}(x,\mu) = \left[\frac{d^j}{dx^j}\right]\left(\sum_{r=0}^{4}(\mu i^{\nu-1})^{-r}\varphi_r(x)\exp(i^{\nu-1\mu x})\right)\exp(-i^{\nu-1\mu x}) + o(\mu^{-4+j}), \tag{1.2.26}$$

$j = 0,1,2,3$ where $\left[\frac{d^j}{dx^j}\right]$ means omission of terms of the Leibniz expansion which contain a function $\varphi_r^{(k)}$ with $k > 4-r$. Since the coefficient of $y^{[3]}$ in (1.2.17) is zero, $\varphi_0(x) = 1$.

The functions φ_1 and φ_2 were determined by

$$\varphi_r = \varepsilon_1^T V Q^{[r]}\varepsilon_1, \tag{1.2.27}$$

where ε_ν is the ν-th unit vector in $\mathbb{C}^4$, $V = (i^{(j-1)(k-1)})_{j,k=1}^4$ and $Q^{[r]}$ are 4×4 matrices given by

$$\Omega_4 Q^{[1]} - Q^{[1]}\Omega_4 = Q^{[0]'} = 0, \tag{1.2.28}$$

$$\Omega_4 Q^{[2]} - Q^{[2]}\Omega_4 = Q^{[1]'} - \frac{1}{4}g\Omega_4\varepsilon\varepsilon^T\Omega_4^{-2}Q^{[0]}, \tag{1.2.29}$$

$$0 = \varepsilon_\nu^T\left(Q^{[2]'} + \frac{1}{4}\sum_{j=1}^{2}k_{3-j}\Omega_4\varepsilon\varepsilon^T\Omega_4^{-1-j}Q^{[2-j]}\right)\varepsilon_\nu, \quad \nu = 1,2,3,4, \tag{1.2.30}$$

where $k_2 = -g$, $k_1 = -g'$, $\Omega_4 = diag(1,i,-1,-i)$, $Q^{[0]} = I_4$ and $\varepsilon^T = (1,1,1,1)$. Let $G(x) = \int_0^x g(t)dt$. After some lengthy calculation, it gives that

$$\varphi_1 = \frac{1}{4}G, \quad \varphi_2 = \frac{1}{32}G^2 - \frac{1}{8}g \tag{1.2.31}$$

and thus

$$\eta_\nu = \Big(1 + \frac{1}{4}i^{-\nu+1}G\mu^{-1} + (-1)^{\nu-1}(\frac{1}{32}G^2 - \frac{1}{8}g)\mu^{-2}\Big)\exp(i^{\nu-1}\mu x)$$
$$+ \{o(\mu^{-2})\}_\infty \exp(i^{\nu-1}\mu x) \tag{1.2.32}$$

for $\nu = 1, 2, 3, 4$ where $\{o()\}_\infty$ means that the estimate is uniform in x. Lastly, we state the following theorem from their paper.

Theorem 1.2.1. *For $g \in C^1[0,a]$, there exists a positive integer k_0 such that the eigenvalues λ_k, $k \in \mathbb{Z}$, counted with multiplicity, of the problem* (1.2.17) – (1.2.18)*, where*

$$B_1(y) = y^{[p_1]}(0), \tag{1.2.33}$$
$$B_2(y) = y^{[p_2]}(0), \tag{1.2.34}$$
$$B_3(y) = y''(a) + i\alpha\lambda y'(a) \tag{1.2.35}$$

and

$$B_4(y) = y^{(3)}(a) - i\alpha\lambda y(a), \tag{1.2.36}$$

can be enumerated in such a way that the eigenvalues λ_k are pure imaginary for $|k| < k_0$ and $\lambda_{-k} = -\bar{\lambda}_k$ for $k \geq k_0$, where k_0 is a positive integer and for $k \geq k_0$, $\lambda_k = \mu_k^2$, where the μ_k have the asymptotics

$$\mu_k = k\frac{\pi}{a} + \tau_0 + \frac{\tau_1}{k} + \frac{\tau_2}{k^2} + o(k^{-2}) \tag{1.2.37}$$

and the numbers τ_0,τ_1,τ_2 are as follows:

Case 1*:* $p_1 = 0$, $p_2 = 1$

$$\tau_0 = -\frac{\pi}{4a}, \tag{1.2.38}$$

$$\tau_1 = \frac{i}{2}\frac{1+\alpha^2}{\pi\alpha} + \frac{1}{4}\frac{G(a)}{\pi}, \tag{1.2.39}$$

$$\tau_2 = \frac{i}{8}\frac{1+\alpha^2}{\pi\alpha} - \frac{a((1-\alpha^2)^2 + \alpha^2 g(0))}{4\pi^2\alpha^2} + \frac{1}{16}\frac{G(a)}{\pi}. \tag{1.2.40}$$

Case 2*:* $p_1 = 0$, $p_2 = 2$

$$\tau_0 = -\frac{\pi}{2a}, \tag{1.2.41}$$

$$\tau_1 = \frac{i}{2}\frac{1+\alpha^2}{\pi\alpha} + \frac{1}{4}\frac{G(a)}{\pi}, \tag{1.2.42}$$

$$\tau_2 = \frac{i}{4}\frac{1+\alpha^2}{\pi\alpha} - \frac{a(1-\alpha^2)^2}{4\pi^2\alpha^2} + \frac{1}{8}\frac{G(a)}{\pi}. \tag{1.2.43}$$

Case 3*:* $p_1 = 1$, $p_2 = 3$

$$\tau_0 = -\frac{\pi}{a}, \tag{1.2.44}$$

$$\tau_1 = \frac{i}{2}\frac{1+\alpha^2}{\pi\alpha} + \frac{1}{4}\frac{G(a)}{\pi}, \tag{1.2.45}$$

$$\tau_2 = \frac{i}{2}\frac{1+\alpha^2}{\pi\alpha} - \frac{a(1-\alpha^2)^2}{4\pi^2\alpha^2} + \frac{1}{4}\frac{G(a)}{\pi}. \tag{1.2.46}$$

Case 4*:* $p_1 = 2$, $p_2 = 3$

$$\tau_0 = -\frac{5\pi}{4a}, \tag{1.2.47}$$

$$\tau_1 = \frac{i}{2}\frac{1+\alpha^2}{\pi\alpha} + \frac{1}{4}\frac{G(a)}{\pi}, \tag{1.2.48}$$

$$\tau_2 = \frac{5i}{8}\frac{1+\alpha^2}{\pi\alpha} - \frac{a((1-\alpha^2)^2 - 3\alpha^2 g(0))}{4\pi^2\alpha^2} + \frac{5}{16}\frac{G(a)}{\pi}. \tag{1.2.49}$$

In particular, there is an odd number of pure imaginary eigenvalues in each case.

1.3 Introduction of the research project

We now turn our focus to the work of this research project. In this research project, we investigate eigenvalue asymptotics associated with boundary conditions leading to the self-adjoint or non-self adjoint operator representations.

We consider the eigenvalue problem,

$$-y'' + gy' + hy = \lambda y, \tag{1.3.1}$$

$$B_j y = 0 \quad j = 1, 2, \tag{1.3.2}$$

on the interval $[0, a]$ where $a > 0$, $h, g \in C^1[0, a]$ are real-valued functions and (1.3.2) are separated and periodic boundary conditions.

In this project, we shall look at two sets of boundary conditions. The separated boundary conditions given by

$$B_j y = \cos\theta_j y(a_j, \lambda) + \sin\theta_j y'(a_j, \lambda), \tag{1.3.3}$$

where $\theta_j = \alpha, a_j = 0$ for $j = 1$ and $\theta_j = \beta, a_j = a$ for $j = 2$ and $\theta_j \in (0, \pi]$,

and the periodic boundary conditions given by

$$B_j y = y^{(i)}(0, \lambda) - y^{(i)}(a, \lambda), \tag{1.3.4}$$

where $i = 0$ for $j = 1$ and $i = 1$ for $j = 2$.

Chapter 2

PRELIMINARIES

In this chapter, we give basic and important definitions and concepts required to undertake and understand the work to be studied. Notions and notations to be introduced in this chapter will be used and assumed in the subsequent chapters.
We start with definitions to build up the understanding of the work to be carried out.

2.1 Banach spaces

Here, we let E to be any non-empty set.

The following definitions are taken from pages 1 and 2 of [19].

Definition 2.1.1. *A linear space E is a set of elements $x_1, x_2, \dots$ for which linear operations are defined and the operations are subject to general rules. That is for which, summation $x_1 + x_2$ of the two elements x_1, x_2, and multiplication λx_1 of the element x_1 by the complex number λ are defined. A zero element is denoted by* 0 *as a zero scalar.*

Definition 2.1.2. *Elements $x_1, \dots, x_n$ in E are called linearly independent if their linear combination $\lambda_1 x_1 + \cdots + \lambda_n x_n$ is equal to zero if and only if $\lambda_1 = \cdots = \lambda_n = 0$.*

Otherwise these elements are linearly dependent.

Definition 2.1.3. *A linear space E is called a linear normed space if each element $x \in E$ is connected to a non-negative number designated $||x||$, which is called the norm of the element x and has the following properties:*
a) $||x|| = 0$ *if and only if* $x = 0$;
b) $||x + y|| \leq ||x|| + ||y||$;
c) $||\lambda x|| = |\lambda| ||x||$.

The following definition is taken from pages 2 and 3 of [12].

Definition 2.1.4. 1. *A sequence $(y_n)_{n \in \mathbb{N}}$ in E is called a Cauchy sequence in a normed vector space E, if for each $\epsilon > 0$ there is a number $n_0 \in \mathbb{N}$ such that $||y_n - y_m|| \leq \epsilon$ for $n, m \geq n_0$.*

2. *A normed space E is called a Banach space if each Cauchy sequence in E is convergent to some $y \in E$*

3. *Let E and F be Banach spaces, then $L(E, F)$ denotes the space of all continuous linear operators on E to F i.e $T \in L(E, F)$ if and only if $T : E \to F$ is linear and*

$$||T|| := \sup\{||Ty||_F : y \in E, ||y||_E \leq 1\} < \infty$$

$L(E, F)$ is a Banach space with norm $||.||$

4. *$E' := L(E, \mathbb{C})$ is called the* dual space *of E. The bilinear form $\langle , \rangle$ on $E \times E'$ is defined by $\langle y, u\rangle = u(y)$ where $y \in E$ and $u \in E'$. With respect to the norm*

$$||u|| := \sup\{||\langle y, u\rangle|| : ||y|| \leq 1\},$$

which is the operator norm on $L(E, \mathbb{C})$, E' is a Banach space .

5. *For T in $L(E, F)$ there is a unique $T^* \in L(F', E')$ such that*

$$\langle Ty, v\rangle = \langle y, T^* y\rangle, \quad y \in E, v \in F'.$$

The operator T^* is called the adjoint of T.

2.2 Holomorphic vector valued functions

Here, we let Ω be an open nonempty subset of $\mathbb{C}$. The following definition is taken from page 6 of [12].

Definition 2.2.1. *Let E be a Banach space, $y : \Omega \to E$, and $\lambda_0 \in \Omega$. The vector function y is called* holomorphic *at λ_0 if there are: a number $r > 0$ and a sequence $(y_n)_{n\in\mathbb{N}}$ in E, such that $K_r(\lambda_0) := \{\lambda \in \mathbb{C} : ||\lambda - \lambda_0|| < r\} \subset \Omega$,*

$$\sum_{n=0}^{\infty} r^n ||y_n|| < \infty \tag{2.2.1}$$

and

$$y(\lambda) = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n y_n \tag{2.2.2}$$

for all $\lambda \in K_r(\lambda_0)$. Because of (2.2.1) the series (2.2.2) is absolutely convergent in E and thus $y(\lambda)$ is well-defined. The vector function y is called holomorphic *in Ω if it is holomorphic at each $\lambda \in \Omega$.*

$H(\Omega, E)$ denotes the set of all holomorphic functions from Ω to E; if E_0 is any subset of E, then

$$H(\Omega, E_0) := \{f \in H(\Omega, E) : f(\Omega) \subset E_0\}.$$

Remark 2.2.2. *A function $f : \Omega \to E$ is holomorphic if and only if it is (continuously) differentiable.*

2.3 Sobolev spaces on intervals

In this section we assume that a and b are real numbers with $a < b$. Furthermore let $1 \leq p \leq \infty$. There is a unique p' such that $1 \leq p' \leq \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

2.3.1 Function spaces

Let $I \subset \mathbb{R}$ be an interval. $C(I) = C^0(I)$ denotes the space of all continuous functions on I to $\mathbb{C}$. For a positive integer k, $C^k(I)$ denotes the space of k-times continuously differentiable functions on I.
Let

$$C^\infty(I) := \bigcap_{k=1}^{\infty} C^k(I).$$

For $f \in C(I)$ the set

$$\mathrm{supp} f := \overline{\{x \in I : f(x) \neq 0\}}$$

is called the *support* of f, where the closure is taken with respect to I. If I is compact, we set

$$||f||_{(k)} := \sum_{j=0}^{k} \max_{x \in I} ||f^{(j)}(x)|| \quad (f \in C^k(I)).$$

We define as usual, $L_p(I)$, the space of measurable functions f on I such that

$$||f||_p := (\int_I |f(x)|^p dx)^{\frac{1}{p}} < \infty \quad (1 \leq p < \infty),$$
$$||f||_\infty := \operatorname{ess\,sup}\{|f(x)| : x \in I\} < \infty \quad (p = \infty).$$

Now let I be open. A function $f \in C^\infty(I)$ is called a test function if its support is a compact subset of I. The space of all test functions on an open interval I is denoted by $C_0^\infty(I)$ which can be identified with a subspace of $C_0^\infty(\mathbb{R})$ by setting

$f = 0$ outside of I for each $f \in C_0^\infty(I)$. Thus

$$C_0^\infty(I) = \bigcup_{K \subset I, \text{compact}} C_0^\infty(K),$$

where

$$C_0^\infty(K) := \{f \in C_0^\infty(\mathbb{R}) : \text{supp} f \subset K\}.$$

A linear functional u on $C_0^\infty(I)$ is called a *distribution* on I if for each compact set $K \subset I$ there are numbers $k \in \mathbb{N}$ and a $C \geq 0$ such that

$$||\langle \varphi, u \rangle|| \leq C \sum_{j=0}^{k} \sup_{x \in K} ||\varphi^{(j)}(x)||, \qquad \varphi \in C_0^\infty(K), \tag{2.3.1}$$

where $\langle \varphi, u \rangle := u(\varphi)$. The space of distributions on I is denoted by $\mathscr{D}'(I)$. For $u \in \mathscr{D}'(I)$ the *support* of u, denoted supp u, is the set of points $x \in I$ such that for each neighbourhood $U \subset I$ of x there is a function $\varphi \in C_0^\infty(U)$ such that $\langle \varphi, u \rangle \neq 0$. For $\Omega \subset \mathbb{R}^n$, $u, v \in L_p(\Omega)$, and α a multi-index, we define the α-th distributional derivative or weak derivative u of v if:

$$\int_U v D^\alpha \varphi dx = (-1)^{|\alpha|} \int_U u \varphi dx$$

for all $\varphi \in C_0^\infty(U)$.

2.3.2 Sobolev spaces

We now introduce *Sobolev spaces*. The *Sobolev spaces*, $W_p^k(I)$, [7] were introduced by Sobolev in 1938. They are important because a number of operator estimates are naturally carried out using them. As k increases functions in $W_p^k(I)$ become better behaved locally, so it is advantageous to know that a function lies in $W_p^k(I)$ for a large value of k.
A function $f \in L_2$ is said to lie in $W_2^k(I)$ for a given positive integer k if the weak partial derivatives $D^\alpha f$ lie in L^2 for all $|\alpha| \leq k$. We then define the Sobolev norm

of such functions by

$$||f||_k^2 := \sum_{|\alpha|\leq k} ||D^\alpha f||_2^2.$$

We formally give a definition of *Sobolev spaces*. The following definition is taken from page 55 of [12].

Definition 2.3.3. *Let $I \subset \mathbb{R}$ be an open interval and $k \in \mathbb{N}$. The space*

$$W_p^k(I) := \{f \in L_p(I) : \forall j \in \{1, ..., k\}, f^{(j)} \in L_p(I)\}$$

is called a **Sobolev space**. *Here the derivatives $f^{(j)}$ are the derivatives in the sense of distributions. For $f \in W_p^k(I)$ we set*

$$||f||_{p,k} := \sum_{j=0}^{k} ||f^{(j)}||_p.$$

We also note that

$$W_p^0 = L_p(I). \tag{2.3.2}$$

We conclude the section by giving two propositions. The following proposition is from page 55 of [12].

Proposition 2.3.4. *Let $I \subset \mathbb{R}$ be an open interval, $\gamma \in \overline{I}$ and $g \in L_p(I)$. Set*

$$G(x) := \int_\gamma^x g(t)dt \quad (x \in \overline{I}). \tag{2.3.3}$$

Then G is continuous on $\overline{I}$ and $G' = g$ in $\mathscr{D}'(I)$.

More precisely, we should write $(G|_I)' = g$. But since a continuous function on $\overline{I}$ is uniquely determined by its values on I, we shall often identify G and $G|_I$.
The following proposition is from page 56 of [12].

Proposition 2.3.5. *[12] Let $I \subset \mathbb{R}$ be an open interval and $k \in \mathbb{N} \setminus \{0\}$.*

1. *Let $f \in L_p(I)$ and $\gamma \in \overline{I}$. Then $f \in W_p^k(I)$ if and only if there are $g \in W_p^{k-1}(I)$ and $c \in \mathbb{C}$ such that*

$$f(x) = c + \int_\gamma^x g(t)dt \quad (x \in I). \tag{2.3.4}$$

In this case, $g = f'$, f has a continuous extension to $\overline{I}$, which we also denote by f, and $c = f(\gamma)$.

2. $$W_p^k(I) \subset C^{k-1}(\overline{I}).$$

2.4 Linear differential operators

The following definition is from page 3 of [14].

Definition 2.4.1. *A linear differential expression is an expression of the form:*

$$l(y) := p_0(x)y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y \tag{2.4.1}$$

where $\frac{1}{p_0(x)}, p_1(x), p_2(x), \ldots, p_n(x)$ are continuous functions on a fixed, finite, interval $[a, b]$ and $y \in C^n[a, b]$.

The following remark taken from page 3 of [14].

Remark 2.4.2. *For every $y \in C^n[a, b]$ the differential expression $l(y)$ is well defined and represents a function which is continuous on $[a, b]$.*

The following definition is from page 3 of [14].

Definition 2.4.3. *If linear combinations*

$$\begin{aligned} B_j(y) &= \alpha_0^j y(a) + \alpha_1^j y'(a) + \cdots + \alpha_{n-1}^j y^{(n-1)}(a) \\ &\quad + \beta_0^j y(b) + \beta_1^j y'(b) + \cdots + \beta_{n-1}^j y^{(n-1)}(b), \quad j = 1, \ldots, m \end{aligned} \tag{2.4.2}$$

of the values of functions y and their first $n-1$ successive derivatives at the boundary points a and b of the interval $[a, b]$ have been specified and the conditions $B_j(y) = 0, j = 1, \ldots, m$, are imposed on the functions $y \in C^n[a, b]$, these conditions which the functions y must satisfy are called boundary conditions.

The following remark is taken from page 50 of [15].

Remark 2.4.4. *If the coefficients $p_k(x), k = 0, ..., n$ of the differential expression $l(y)$, have continuous derivatives up to the order $(n-k)$ inclusive on the interval $[a, b]$, then there exists a differential expression $l^*(z)$, where $z \in C^n[a, b]$, such that*

$$\int_a^b l(y)\bar{z}dx = \int_a^b y\overline{l^*(z)}dx + [y, z]_a^b, \tag{2.4.3}$$

where

1. $$[y, z] = \sum_{k=1}^{n} (y^{[k-1]}\bar{z}^{[2n-k]} - y^{[2n-k]}\bar{z}^{[k-1]}) \tag{2.4.4}$$

   is the Lagrange's form,

2. $$[y, z]_a^b = [y, z]|_b - [y, z]|_a. \tag{2.4.5}$$

3. (2.4.3) *is said to be Lagrange's identity in integral form.*

The following definition is taken from page 10 of [14].

Definition 2.4.5. *The differential expression $l^*(z)$, defined in* (2.4.3) *is called the adjoint of the differential expression $l(y)$.*

The following definition is from page 10 of [14].

Definition 2.4.6. *If $l = l^*$, then $l(y)$ is said to be formally self-adjoint.*

The following remark is from page 8 of [14].

Remark 2.4.7. *Any self-adjoint differential expression with real coefficients defined on an interval $[a, b]$ is necessarily of even order and has the form*

$$l(y) = (p_0 y^{(q)})^{(q)} + (p_1 y^{(q-1)})^{(q-1)} + ... + (p_{q-1} y')' + p_q y \tag{2.4.6}$$

where $p_0, p_1, ..., p_q$ are real-valued functions, $\frac{1}{p_0(x)} \in C^q[a, b]$ and $p_j \in C^{q-j}[a, b]$ for $j = 1, ..., q$.

In the following definition, taken from page 3 and 13 of [14], we define the operator L, to be the differential operator generated by a differential expression $l(y)$ and the boundary conditions $B_j(y) = 0, j = 0, \ldots, m$.

Definition 2.4.8. *A number λ is called an* **eigenvalue** *of an operator L, generated by a differential expression $l(y) = \lambda y$ and the boundary conditions $B_1(y), ..., B_m(y)$, if there exists in the domain of definition of the operator L a function y, not identical to zero, such that $Ly = \lambda y$. The function y is called an* **eigenfunction** *of the operator L for the eigenvalue λ.*

2.5 Fundamental matrices

Let Ω be a nonempty open subset of $\mathbb{C}$, $n \in \mathbb{N} \setminus \{0\}$ and $A \in H(\Omega, M_n(L_p(a,b)))$, where $M_n(L_p(a,b))$ is the set of $n \times n$ matrices with entries from $L_p(a,b)$. The value of the matrix function A at $\lambda \in \Omega$ and $x \in (a,b)$ is denoted by $A(x,\lambda)$.
We set

$$T^D(\lambda)y := y' - A(.,\lambda)y, \quad y \in (W_p^1(a,b))^n, \lambda \in \Omega \tag{2.5.1}$$

The following definition is from page 69 of [12].

Definition 2.5.1. *Let $\lambda_0 \in \Omega$. A matrix $Y_0 \in M_n(W_p^1(a,b))$ is called a* fundamental matrix *of $T^D(\lambda_0)y = 0$ if for each $y \in N(T^D(\lambda_0))$ there is a $c \in \mathbb{C}^n$ such that $y = Y_0 c$. A matrix function $Y : \Omega \to M_n(W_p^1(a,b))$ is called a* fundamental matrix function *of $T^D y = 0$ if $Y(\lambda)$ is a fundamental matrix of $T^D(\lambda)y = 0$ for each $\lambda \in \Omega$.*

2.6 Asymptotic behaviour of functions

The following definitions are taken from page 76 of [12]. Let U be an unbounded subset of $\mathbb{C}$ and $M_{k,n}(\mathbb{C})$ be a set of $k \times n$ matrices with entries from $\mathbb{C}$.

Definition 2.6.1. *Let f be a function on U with values in $M_{k,n}(\mathbb{C})$ and g be a complex-valued function on U. We write*

$$f(\lambda) = O(g(\lambda))$$

if there is a $C > 0$ such that $|f(\lambda)| \le C|g(\lambda)|$ for $\lambda \in U$. The notation

$$f(\lambda) = o(g(\lambda))$$

means that $|f(\lambda)||g(\lambda)|^{-1} \to 0$ as $\lambda \to \infty$ in U. Let $a \in M_{k,n}(\mathbb{C})$. We write

$$f(\lambda) = [a]$$

if $f(\lambda) - a = o(1)$.

Definition 2.6.2. *Let* $f(.,\lambda) \in M_{k,n}(L_p(a,b))$ *for* $\lambda \in U$, g *be a complex-valued function on* U. *We write*

$$f(.,\lambda) = O(g(\lambda))_p \quad or \quad f(.,\lambda) = O(g(\lambda)) \quad in \quad M_{k,n}(L_p(a,b))$$

if there is a $C > 0$ *such that* $|f(.,\lambda)|_p \leq C|g(\lambda)|$ *for* $\lambda \in U$, *and*

$$f(.,\lambda) = o(g(\lambda))_p \quad or \quad f(.,\lambda) = o(g(\lambda)) \quad in \quad M_{k,n}(L_p(a,b))$$

if $|f(.,\lambda)|_p|g(\lambda)|^{-1} \to 0$ *as* $\lambda \to \infty$. *For* $h \in M_{k,n}(L_p(a,b))$, *we write*

$$f(.,\lambda) = [h]_p$$

if $f(.,\lambda) - h = o(1)_p$.

2.7 n-th order differential equations and systems

In this section we again let Ω be a nonempty open subset of $\mathbb{C}$, $-\infty < a < b < \infty$, $1 \leq p \leq \infty$, $1 \leq p' \leq \infty$ such that $\frac{1}{p} + \frac{1}{p'} = 1$, and $n \in \mathbb{N}$, $n \geq 2$. By e_j we denote the j-th unit vector in $\mathbb{C}^n$. We consider the scalar n-th order differential equation

$$\eta^{(n)} + \sum_{i=0}^{n-1} p_i(.,\lambda)\eta^{(i)} = 0 \quad (\eta \in W_p^n(a,b), \lambda \in \Omega), \tag{2.7.1}$$

where $p_i \in H(\Omega, L_p(a,b)) \quad (i = 0, ..., n-1)$. Together with this differential equation we consider the differential operator

$$L^D(\lambda) := \eta^{(n)} + \sum_{i=0}^{n-1} p_i(.,\lambda)\eta^{(i)} \quad (\eta \in W_p^n(a,b), \lambda \in \Omega). \tag{2.7.2}$$

We associate the first order system to the n-th order differential equation. The n-th order differential system is defined by the operator

$$T^D(\lambda)y := y' - A(.,\lambda)y, \quad y \in (W_p^1(a,b))^n, \lambda \in \mathbb{C}, \tag{2.7.3}$$

where

$$A := (\delta_{i,j-1} - \delta_{i,n}p_{j-1})_{i,j=1}^n = \begin{pmatrix} 0 & 1 & & & \\ & . & . & 0 & \\ & . & . & & \\ & 0 & 0 & & 1 \\ -p_0 & . & . & . & -p_{n-1} \end{pmatrix}. \tag{2.7.4}$$

We take the following proposition from page 251 of [12].

Proposition 2.7.1. *Let* $\eta \in W_p^n(a,b)$, $\lambda \in \Omega$, *and set*

$$y := \begin{pmatrix} \eta \\ \eta' \\ . \\ . \\ . \\ \eta^{(n-1)} \end{pmatrix}.$$

Then $y \in (W_p^1(a,b))^n$ *and*

$$T^D(\lambda)y = \begin{pmatrix} 0 \\ . \\ . \\ . \\ 0 \\ L^D(\lambda)\eta \end{pmatrix}.$$

Proof. The assertions $y \in (W_p^1(a,b))^n$ and

$$e_i^\top A(.,\lambda) = e_{i+1}^\top, \quad i = 1, \ldots, n-1 \tag{2.7.5}$$

are trivial. From (2.7.3) and (2.7.5) we have that for $i = 1, \ldots, n-1$

$$\begin{aligned} e_i^\top T^D(\lambda)y &= e_i^\top y' - e_i^\top A(., \lambda)y \\ &= \eta^{(i)} - e_{i+1}^\top y \\ &= 0. \end{aligned}$$

For the n-th entry and from (2.7.2) we have that

$$\begin{aligned} e_n^\top T^D(\lambda)y &= e_n^\top y' + \sum_{i=0}^{n-1} p_i(., \lambda) e_{i+1}^\top y \\ &= \eta^{(n)} + \sum_{i=0}^{n-1} p_i(., \lambda)\eta^{(i)} \\ &= L^D(\lambda)\eta. \end{aligned}$$

Thus

$$T^D(\lambda)y = \begin{pmatrix} 0 \\ . \\ . \\ . \\ 0 \\ L^D(\lambda)\eta \end{pmatrix}.$$

□

The following proposition is from page 252 of [12].

Proposition 2.7.2. *Let $y \in (W_p^1(a,b))^n$, $\lambda \in \Omega$, and assume that $e_i^\top T^D(\lambda)y = 0$*

for $i = 1, ..., n-1$. *Then* $\eta := e_1^\top y \in W_p^n(a,b)$,

$$y = \begin{pmatrix} \eta \\ \eta' \\ \cdot \\ \cdot \\ \cdot \\ \eta^{(n-1)} \end{pmatrix} \tag{2.7.6}$$

and

$$L^D(\lambda)\eta = e_n^\top T^D(\lambda)y. \tag{2.7.7}$$

Proof. We show the first part of the proposition by induction. Let $i \in \{1, \ldots, n-1\}$. Since $e_i^\top T^D(\lambda)y = 0$ and from Proposition 2.7.1 we get

$$e_i^\top y' = e_i^\top y' - e_i^\top T^D y = e_i^\top y' - (e_i^\top y' - e_i^\top A(.,\lambda)y) = e_i^\top A(.,\lambda)y = e_{i+1}^\top y. \tag{2.7.8}$$

It proves that $\eta \in W_p^i(a,b)$ and $e_i^\top y = \eta^{(i-1)}$ for $i = 1, \ldots, n$. This is true for $i = 1$. We now assume that it holds for some $i < n$ and show that it is true for $i+1$. From the induction assumption, (2.7.8) yields

$$\eta^{(i)} = e_i^\top y' = e_{i+1}^\top y \in W_p^1(a,b).$$

By Corollary 2.1.4 of [12], we have that $\eta \in W_p^{i+1}(a,b)$. Thus $\eta \in W_p^n(a,b)$ and (2.7.6) holds. Thus (2.7.7) follows from Proposition 2.7.1 and equation (2.7.6). $\square$

We take the following definition from page 252 of [12].

Definition 2.7.3. *Let* $\lambda_0 \in \Omega$ *and* $\eta_1, ..., \eta_n \in W_p^n(a,b)$. *Then* $\{\eta_1, ..., \eta_n\}$ *is called a* fundamental system *of* $L^D(\lambda_0)\eta = 0$ *if for each* $\eta \in N(L^D(\lambda_0))$ *there are* $c_j \in \mathbb{C}$, $j = 1, ..., n$ *such that*

$$\eta = \sum_{j=1}^n c_j \eta_j.$$

A function $(\eta_1, ..., \eta_n) : \Omega \to M_{1,n}(W_p^n(a,b))$ *is called a* fundamemtal system function *of $L^D y = 0$ if $\{\eta_1(\lambda), ..., \eta_n(\lambda)\}$ is a fundamental system of $L^D(\lambda)y = 0$ for each $\lambda \in \Omega$.*

The following lemma is taken from page 253 of [12].

Lemma 2.7.4. *Let $\lambda_0 \in \Omega$ and $Y_0 \in M_n(W_p^1(a,b))$ be a fundamental matrix of $T^D(\lambda_0)y$. Then $\{e_1^\top Y_0 e_1, ..., e_1^\top Y_0 e_n\}$ is a fundamental system $L^D(\lambda_0)\eta = 0$, and*

$$(e_1^\top Y_0 e_j)^{(i-1)} = e_i^\top Y_0 e_j \tag{2.7.9}$$

holds for $i = 1, ..., n$ and $j = 1, ..., n$

The following lemma is from page 253 of [12].

Lemma 2.7.5. *Let $\lambda_0 \in \Omega$ and $\eta_1, ..., \eta_n \in W_p^n(a,b)$ such that $\{\eta_1, ..., \eta_n\}$ is a fundamental system of $L^D(\lambda_0) = 0$. Then $(\eta_j^{(i-1)})_{i,j=1}^n \in M_n(W_p^1(a,b))$ is a fundamental matrix of $T^D(\lambda_0)y = 0$.*

Proof. Let $y \in N(T^D(\lambda_0))$. From Proposition 2.7.2 we have that $\eta := e_1^\top y \in W_p^n(a,b)$,

$$y = \begin{pmatrix} \eta \\ \eta' \\ . \\ . \\ . \\ \eta^{(n-1)} \end{pmatrix}$$

and $L^D(\lambda_0)\eta = e_n^\top T^D(\lambda_0)y = 0$. Hence by the definition above there is a vector $c = (c_1, \ldots, c_n)^\top \in \mathbb{C}^n$ such that

$$\eta = \sum_{j=1}^n c_j \eta_j.$$

Thus $y = (\eta_j^{(i-1)})_{i,j=1}^n c$ is a fundamental matrix of $T^D(\lambda_0)y = 0$. $\square$

The proposition is from page 253 of [12].

Proposition 2.7.6. *Let $\lambda_0 \in \Omega$ and $\eta_1, ..., \eta_n \in W_p^n(a,b)$. Then the following conditions are equivalent:*

1. *$\eta_1, ..., \eta_n$ are linearly independent, $L^D(\lambda_0)\eta_j = 0$ for each $j \in \{1, ..., n\}$,and for each $\eta \in N(L^D(\lambda_0))$ there are $c_j \in \mathbb{C}$ $(j = 1, ..., n)$ such that*

$$\eta = \sum_{j=1}^{n} c_j \eta_j;$$

2. *$\{\eta_1, ..., \eta_n\}$ is a fundamental system of $L^D(\lambda_0)\eta = 0$;*

3. *$(\eta_j^{(i-1)})_{i,j=1}^n$ is a fundamental matrix of $T^D(\lambda_0)y = 0$.*

The following theorem is from page 254 of [12].

Theorem 2.7.1. *There is a fundamental system function $(\eta_1, ..., \eta_n)$ of $L^D\eta = 0$ such that $\eta_j^{(i-1)}(a, \lambda) = \delta_{i,j}$ for $\lambda \in \Omega$ and $i, j = 1, ..., n$. Furthermore, the fundamental system function is uniquely determined and depends holomorphically on $\lambda \in \Omega$. More precisely, we have $\eta_j \in H(\Omega, W_p^n(a, b))$ for $j = 1, ..., n$.*

2.8 The boundary eigenvalue operator function

We let $L^R \in H(\Omega, L(W_p^n(a, b), \mathbb{C}^n))$ and L^D be defined as in (2.7.2). We call

$$L = (L^D, L^R) \in H(\Omega, L(W_p^n(a, b), L_p(a, b) \times \mathbb{C}^n)) \tag{2.8.1}$$

a *boundary eigenvalue operator function.*

Let $\{\eta_1, ..., \eta_n\}$ be the fundamental system function of $L^D\eta = 0$ given by Theorem

2.7.1 and set $Y := (\eta_j^{(i-1)})_{i,j=1}^n$. Define

$$Z_L(\lambda)c := (\eta_1(.,\lambda), ..., \eta_n(.,\lambda))c = e_1^\top Y(.,\lambda)c \quad (c \in \mathbb{C}^n, \lambda \in \Omega) \tag{2.8.2}$$

and

$$(U_L(\lambda)f)(x) := e_1^\top Y(x,\lambda) \int_a^x Y(t,\lambda)^{-1} e_n f(t)dt \quad (f \in L_p(a,b)). \tag{2.8.3}$$

From Lemma 2.7.5 we know that Y is a fundamental matrix function of $T^D y = 0$. Let $\lambda \in \Omega$ and $U(\lambda)$ be the right inverse of $T^D(\lambda)$ given by

$$(U_L(\lambda)f)(x) = Y(x,\lambda) \int_a^x Y(t,\lambda)^{-1} f(t)dt \tag{2.8.4}$$

for $\lambda \in \Omega$, $f \in (L_p(a,b))^n$ and $x \in (a,b)$.
Then

$$U_L(\lambda) = e_1^\top U(\lambda) e_n. \tag{2.8.5}$$

The characteristic matrix function of L is defined by

$$M(\lambda) = L^R(\lambda) Z_L(\lambda). \tag{2.8.6}$$

The theorem is from page 259 of [12].

Theorem 2.8.1. *The boundary eigenvalue operator function L given by* (2.8.1) *is holomorphically equivalent on Ω to the $L_p(a,b)$-extension of M; more precisely, for $\lambda \in \Omega$ we have*

$$L(\lambda) = \begin{pmatrix} 0 & id_{L_p(a,b)} \\ id_{\mathbb{C}^n} & L^R(\lambda)U_L(\lambda) \end{pmatrix} \begin{pmatrix} M(\lambda) & 0 \\ 0 & id_{L_p(a,b)} \end{pmatrix} (Z_L(\lambda), U_L(\lambda))^{-1},$$

and the operators

$$\begin{pmatrix} 0 & id_{L_p(a,b)} \\ id_{\mathbb{C}^n} & L^R(\lambda)U_L(\lambda) \end{pmatrix} \in L(\mathbb{C}^n \times L_p(a,b), L_p(a,b) \times \mathbb{C}^n)$$

and

$$(Z_L(\lambda), U_L(\lambda)) \in L(\mathbb{C}^n \times L_p(a,b), W_p^n(a,b)))$$

are invertible and depend holomorphically on λ*.*

2.9 Two-point boundary eigenvalue problems in $L_p(a,b)$

In this section, we let $p < \infty$ and

$$L^D(\lambda)\eta = \eta^{(n)} + \sum_{i=0}^{n-1} p_i(.,\lambda)\eta^{(i)},$$

$$L^R(\lambda)\eta = W^a(\lambda)\begin{pmatrix} \eta(a) \\ . \\ . \\ . \\ \eta^{(n-1)}(a) \end{pmatrix} + W^b(\lambda)\begin{pmatrix} \eta(b) \\ . \\ . \\ . \\ \eta^{(n-1)}(b) \end{pmatrix}$$

for $\lambda \in \Omega$ and $\eta \in W_p^n(a,b)$, where $p_i \in H(\Omega, W^i_{\max\{p,p'\}}(a,b))$ $(1 \leq i \leq n-1)$ and $W^a, W^b \in H(\Omega, M_n(\mathbb{C}))$. From equation (2.7.1) we know that $p_0 \in H(\Omega, L_p(a,b))$ and this is substantiated by equation (2.3.2).
We suppose that $\operatorname{rank}(W^a(\lambda), W^b(\lambda)) = n$ for all $\lambda \in \Omega$.

Apart from L^D we consider $L^{D^+} \in H(\Omega, L(W^n_{p'}(a,b), L_{p'}(a,b))$ defined by

$$L^{D^+}(\lambda)\eta = (-1)^n\eta^{(n)} + \sum_{i=0}^{n-1}(-1)^i(p_i(.,\lambda)\eta)^{(i)}$$

for $\lambda \in \Omega$ and $\eta \in W^n_{p'}(a,b)$. We define $L_0(\lambda)$ in $L_p(a,b)$ by

$$D(L_0(\lambda)) = \{\eta \in W^n_p(a,b) : L^R(\lambda)\eta = 0\} \subset L_p(a,b) \tag{2.9.1}$$

and

$$L_0(\lambda)\eta = L^D(\lambda)\eta, \quad \eta \in D(L_0(\lambda)). \tag{2.9.2}$$

The following definition is from page 275 of [12].

Definition 2.9.1. *Let $\eta \in H(\Omega, W^n_p(a,b))$ and $\mu \in \Omega$. η is called a* root function *of L_0 at μ if and only if $\eta(\mu) \neq 0$, $(L^D\eta)(\mu) = 0$ and $W^a(\mu)(\eta^{(i)}(a,\mu))_{i=0}^{n-1} + W^b(\mu)(\eta^{(i)}(b,\mu))_{i=0}^{n-1} = 0$. The minimum of the orders of the zero of $L^D\eta$ and $W^a(\eta^{(i)}(a,.))_{i=0}^{n-1} + W^b(\eta^{(i)}(b,.))_{i=0}^{n-1}$ at μ is called the* multiplicity *of η.*

From $L^R\eta = W^a(\eta^{(i)}(a,.))_{i=0}^{n-1} + W^b(\eta^{(i)}(b,.))_{i=0}^{n-1}$ we obtain the following Proposition, taken from page 275 of [12]

Proposition 2.9.2. *Let $\eta \in H(\Omega, W^n_p(a,b))$, $\mu \in \Omega$ and $\nu \in \mathbb{N}$. Then η is a root function of L_0 of multiplicity ν at μ if and only if η is a root function of L of multiplicity ν at μ.*

Canonical systems of root functions of L_0 are defined in the same way as for L. Hence a system of root functions is a canonical system of root functions of L_0 at μ if and only if it is a canonical system of root functions of L at μ.

Chapter 3

ASYMPTOTICS OF EIGENVALUES FOR $g, h = 0$

In this chapter we consider the eigenvalue problem (1.3.1)-(1.3.2) with $g, h = 0$. As shown in [13], putting

$$M(\lambda) = (B_i y_j(.,\lambda))_{i,j=1}^2, \tag{3.0.1}$$

the eigenvalues of the eigenvalue problem (1.3.1)-(1.3.2) are the eigenvalues of the analytic matrix function M. For $g, h = 0$, (1.3.1) reduces to $y'' + \lambda y = 0$. Let $y = e^{\rho x}$. Then the equation becomes $(\rho^2 + \lambda)e^{\rho x} = 0$. Hence the characteristic function, (see, page 280 of [12]) of the differential equation for $g, h = 0$ is

$$\pi(\mu, \rho) = \rho^2 + \mu^2 \tag{3.0.2}$$

where $\lambda = \mu^2$. The zeros of (3.0.2) are $i\mu, -i\mu$ and thus a fundamental system of $y'' + \lambda y = 0$ is $\{e^{i\mu x}, e^{-i\mu x}\}$ if $\mu \neq 0$ and the fundamental matrix is

$$Z(x, \mu) = \begin{pmatrix} e^{i\mu x} & e^{-i\mu x} \\ i\mu e^{i\mu x} & -i\mu e^{-i\mu x} \end{pmatrix}.$$

Let $y_j, j = 1, 2$ be the fundamental system of the differential equation (1.3.1) which is analytic on $\mathbb{C}$ with respect to λ with $I_2 = Y(0, \mu)$, (see Theorem 2.5.3 of [12]) where $Y = (y_j^{(i-1)})_{i,j=1}^2$. Then there is $C \in M_2(\mathbb{C})$ such that $Y(x,\mu) = Z(x,\mu)C$, (see Definition 2.5.1) and hence $C = Z(0,\mu)^{-1}$. From the above fundamental matrix we have,

$$Z(0,\mu) = \begin{pmatrix} 1 & 1 \\ i\mu & -i\mu \end{pmatrix}.$$

Thus

$$C = -\frac{1}{2i\mu}\begin{pmatrix} -i\mu & -1 \\ -i\mu & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2i\mu} \\ \frac{1}{2} & -\frac{1}{2i\mu} \end{pmatrix}.$$

Therefore

$$\begin{aligned} Y(x,\mu) &= \begin{pmatrix} e^{i\mu x} & e^{-i\mu x} \\ i\mu e^{i\mu x} & -i\mu e^{-i\mu x} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2i\mu} \\ \frac{1}{2} & -\frac{1}{2i\mu} \end{pmatrix} \\ &= \begin{pmatrix} \frac{e^{i\mu x}+e^{-i\mu x}}{2} & \frac{e^{i\mu x}-e^{-i\mu x}}{2i\mu} \\ \frac{i\mu e^{i\mu x}-i\mu e^{-i\mu x}}{2} & \frac{i\mu e^{i\mu x}+i\mu e^{-i\mu x}}{2i\mu} \end{pmatrix} \end{aligned}$$

and it follows that,

$$y_1(x,\mu) = \frac{e^{i\mu x}+e^{-i\mu x}}{2} = \cos\mu x$$

and

$$y_2(x,\mu) = \frac{e^{i\mu x}-e^{-i\mu x}}{2i\mu} = \frac{\sin\mu x}{\mu}.$$

We have that $\det M = \phi_0(\mu)$, where

$$\phi_0(\mu) = \det\begin{pmatrix} B_1 y_1(0,\mu) & B_1 y_2(0,\mu) \\ B_2 y_1(a,\mu) & B_2 y_2(a,\mu) \end{pmatrix}. \tag{3.0.3}$$

In the next section we perform the analysis of the first set of the boundary conditions, that is the boundary conditions in equation (1.3.3).

3.1 Estimates of the eigenvalues with the boundary conditions $\cos\alpha y(\mu,0)+\sin\alpha y'(\mu,0)=0$ and $\cos\beta y(\mu,a)+\sin\beta y'(\mu,a)=0$

We calculate the determinant ϕ_0 of the matrix M. The entries of the matrix M are computed as follows:

$$\begin{aligned}
B_1y_1(0,\mu) &= \cos\alpha y_1(\mu,0)+\sin\alpha y_1'(\mu,0) = \cos\alpha,\\
B_1y_2(0,\mu) &= \cos\alpha y_2(\mu,0)+\sin\alpha y_2'(\mu,0) = \sin\alpha,\\
B_2y_1(a,\mu) &= \cos\beta y_1(\mu,a)+\sin\beta y_1'(\mu,a) = \cos\beta\cos\mu a-\mu\sin\beta\sin\mu a,\\
B_2y_2(a,\mu) &= \cos\beta y_2(\mu,a)+\sin\beta y_2'(\mu,a) = \frac{1}{\mu}\cos\beta\sin\mu a+\sin\beta\cos\mu a,
\end{aligned}$$

and thus

$$\begin{aligned}
\phi_0(\mu) &= \cos\alpha(\frac{1}{\mu}\cos\beta\sin\mu a+\sin\beta\cos\mu a)-\sin\alpha(\cos\beta\cos\mu a-\mu\sin\beta\sin\mu a)\\
&= \mu\sin\mu a(\frac{1}{\mu^2}\cos\alpha\cos\beta+\sin\alpha\sin\beta)+\cos\mu a(\cos\alpha\sin\beta-\sin\alpha\cos\beta).
\end{aligned} \tag{3.1.1}$$

We shall equate $\phi_0(\mu)$ to zero, to get the zeros of $\phi_0(\mu)$ on the positive real axis. We can see that the zeros of ϕ_0 cannot be determined explicitly, hence we find ϕ_0 implicitly. We start by the following proposition.

Proposition 3.1.1. *Let $\alpha,\beta\in(0,\pi)$. Then $\xi=\frac{1}{\mu^2}\cos\alpha\cos\beta+\sin\alpha\sin\beta\neq 0$ for sufficiently large μ.*

Proof. We prove this proposition by contradiction.
Suppose $\xi=0$.

Then for $\alpha, \beta \in (0, \pi)$ and large μ

$$\begin{aligned}\frac{1}{\mu^2} &= -\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}\\ \mu^2 &= -\frac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta}\\ &= -\cot\alpha\cot\beta.\end{aligned}$$

Hence a contradiction, since μ is sufficiently large and cannot only be two (complex) numbers. □

Proposition 3.1.2. *Let $\alpha, \beta \in (0, \pi)$ and ϕ_0 be as defined by* (3.1.1)*.*
Then $\cos\mu a \neq 0$ when $\phi_0(\mu) = 0$ and μ sufficiently large.

Proof.
The proposition is proved by contradiction.
Suppose $\cos\mu a = 0$.
Then $\mu \neq 0$ and $\phi_0(\mu) = 0$ implies that

$$\mu\sin\mu a(\frac{1}{\mu^2}\cos\alpha\cos\beta + \sin\alpha\sin\beta) = 0.$$

However, $\cos\mu a = 0$ implies that $\sin\mu a \neq 0$ since $\sin^2\mu a + \cos^2\mu a = 1$. Therefore $\phi_0(\mu) \neq 0$, which is a contradiction. Hence $\phi_0(\mu) = 0$ implies that $\cos\mu a \neq 0$. □

Proposition 3.1.3. *Let $\alpha, \beta \in (0, \pi)$ and ϕ_0 be as defined by* (3.1.1)*. Then there is one simple zero $\tilde{\mu}_k = \frac{\pi}{a}k + o(1), k = 1, 2, \ldots$ of ϕ_0 in each interval $((k-\frac{1}{2})\frac{\pi}{a}, (k+\frac{1}{2})\frac{\pi}{a})$ for sufficiently large positive $k \in \mathbb{Z}$.*

Proof.
Since $\cos\mu a \neq 0$ from Proposition 3.1.2 above, the positive zeros of $\phi_0(\mu)$ are those $\mu > 0$ for which

$$\frac{\cos\alpha\sin\beta - \sin\alpha\cos\beta}{\cos\alpha\cos\beta + \mu^2\sin\alpha\sin\beta}\mu + \frac{\sin\mu a}{\cos\mu a} = 0,$$

that is,

$$\psi(\mu) = \tan \mu a + \frac{\epsilon\mu}{\cos\alpha\cos\beta + \mu^2 \sin\alpha\sin\beta} = 0$$

where $\epsilon = \cos\alpha\sin\beta - \sin\alpha\cos\beta$.

We know that $\tan \mu a$ has period $\frac{\pi}{a}$ with asymptotes at $\mu = \frac{\pi}{2a} \pm k\frac{\pi}{a}$. Since $\tan' x \geq 1$ for $x \in ((k-\frac{1}{2})\frac{\pi}{a}, (k+\frac{1}{2})\frac{\pi}{a}), k \in \mathbb{Z}$, the function $\mu \mapsto \tan \mu a$ is increasing with positive derivative on each interval $((k-\frac{1}{2})\frac{\pi}{a}, (k+\frac{1}{2})\frac{\pi}{a}), k \in \mathbb{Z}$.
The derivative

$$\begin{aligned}\phi_1'(\mu) &= \frac{\epsilon(\cos\alpha\cos\beta - \mu^2\sin\alpha\sin\beta)}{(\cos\alpha\cos\beta + \mu^2\sin\alpha\sin\beta)^2}\\ &= \frac{\epsilon(\frac{1}{\mu^2}\cos\alpha\cos\beta - \sin\alpha\sin\beta)}{\frac{1}{\mu^2}\cos^2\alpha\cos^2\beta + 2\cos\alpha\cos\beta\sin\alpha\sin\beta + \mu^2\sin^2\alpha\sin^2\beta}\end{aligned}$$

where

$$\phi_1(\mu) = \frac{\frac{\epsilon}{\mu}}{\frac{1}{\mu^2}\cos\alpha\cos\beta + \sin\alpha\sin\beta}$$

tends to 0 as $\mu \to \infty$. Hence $\psi'(\mu) > 0$ for large μ and for large k on each of the intervals $((k-\frac{1}{2})\frac{\pi}{a}, (k+\frac{1}{2})\frac{\pi}{a})$, thus the function ψ moves from $-\infty$ to ∞ on each of the intervals $((k-\frac{1}{2})\frac{\pi}{a}, (k+\frac{1}{2})\frac{\pi}{a})$. Thus there is exactly one simple zero $\tilde{\mu}_k$ of ψ in each interval $((k-\frac{1}{2})\frac{\pi}{a}, (k+\frac{1}{2})\frac{\pi}{a})$, for sufficiently large positive $k \in \mathbb{Z}$. From Proposition 3.1.1 we know that $\frac{1}{\mu^2}\cos\alpha\cos\beta + \sin\alpha\sin\beta$ is non-zero for large μ, thus $\phi_1(\mu) \to 0$ as $\mu \to \infty$ and we have that $\arctan(\phi_1(\mu)) \to 0$ as $\mu \to \infty$. Thus we have

$$\tilde{\mu}_k = \frac{\pi}{a}k + o(1), \quad k = 1, 2, \ldots.$$

$\square$

Theorem 3.1.1. *For $g, h = 0$, there exists a positive integer k_0 such that the eigenvalues $\hat{\lambda}_k$, $k \in \mathbb{Z}$, of the problem* (1.3.1)-(1.3.2)*, where $B_1 y = \cos\alpha y(0) + \sin\alpha y'(0)$ and $B_2 y = \cos\beta y(a) + \sin\beta y'(a)$ can be enumerated in such a way that $\hat{\lambda}_k = \hat{\mu}_k^2$, for $k \geq k_0$, where the $\hat{\mu}_k$ have the following estimates:*

$$\hat{\mu}_k = \frac{\pi}{a} k + o(1).$$

3.2 Estimates of the eigenvalues with the boundary conditions $y(\mu, 0) - y(\mu, a) = 0$ and $y'(\mu, 0) - y'(\mu, a) = 0$

We now look at the second set of the boundary conditions, that is the periodic boundary conditions (1.3.4). The entries of the matrix M are as follows:

$$
\begin{aligned}
B_1 y_1(\mu, a) &= y_1(\mu, 0) - y_1(\mu, a) = 1 - \cos \mu a, \\
B_2 y_1(\mu, a) &= y_1'(\mu, 0) - y_1'(\mu, a) = \mu \sin \mu a, \\
B_1 y_2(\mu, a) &= y_2(\mu, 0) - y_2(\mu, a) = -\frac{\sin \mu a}{\mu}, \\
B_2 y_2(\mu, a) &= y_2'(\mu, 0) - y_2'(\mu, a) = 1 - \cos \mu a
\end{aligned}
$$

and from equation (3.0.3) we have that,

$$
\begin{aligned}
\phi_0(\mu) &= (1 - \cos \mu a)(1 - \cos \mu a) - (-\frac{\sin \mu a}{\mu})(\mu \sin \mu a) \\
&= 1 - 2\cos \mu a + \cos^2 \mu a + \sin^2 \mu a \\
&= 2(1 - \cos \mu a).
\end{aligned}
$$

Thus the zeros of ϕ_0 are double zeros given by

$$
\frac{2\pi}{a} k, \quad k \in \mathbb{Z}.
$$

Theorem 3.2.1. *For $g, h = 0$, the eigenvalues $\hat{\lambda}_k$, $k \in \mathbb{Z}$, of the problem* (1.3.1)-(1.3.2)*, where $B_1 y = y(0) - y(a)$ and $B_2 y = y'(0) - y'(a)$ can be enumerated in such a way that $\hat{\lambda}_0 = 0$ and $\hat{\lambda}_k = \hat{\mu}_k^2$, where the $\hat{\mu}_k$ are the following:*

$$\tilde{\mu}_{2k} = \frac{2\pi}{a} k \quad k = 1, 2, \ldots$$
$$\tilde{\mu}_{2k-1} = \frac{2\pi}{a} k \quad k = 1, 2, \ldots$$

The estimates of the eigenvalues under the separate and periodic boundary conditions conclude the chapter. We therefore in the following chapter compute the asymptotics of eigenvalues for general functions g, h, that is calculating the first four terms of the eigenvalue asymptotic.

Chapter 4

THE DIFFERENTIAL EQUATION $K\eta = \lambda H\eta$

4.1 The eigenvalue problem and general assumptions

In this short section we consider the general eigenvalue problem and important assumptions from [12] that are the fundamental building blocks of our work. We will apply the results later in the dissertation for our particular case.
Let $1 \leq p \leq \infty$ and p' such that $\frac{1}{p'} + \frac{1}{p} = 1$. Let $-\infty < a < b < \infty$ and $n \in \mathbb{N}$, $n \geq 2$. We consider the differential equation

$$K\eta = \lambda H\eta, \quad \eta \in W_p^n(a,b), \tag{4.1.1}$$

where

$$K\eta = \eta^{(n)} + \sum_{i=0}^{n-1} k_i \eta^{(i)}, \tag{4.1.2}$$

$$H\eta = \sum_{i=0}^{n_0} h_i \eta^{(i)}, \tag{4.1.3}$$

with $0 \leq n_0 \leq n-1$ and $k_i, h_i \in W_{p'}^i(a,b)$. We shall always assume that $h_{n_0} > 0$

and that $h_{n_0}^{-1} \in L_\infty(a,b)$.

We associate the differential operator

$$L^D(\lambda)\eta := K\eta - \lambda H\eta, \quad \eta \in W_1^n(a,b) \tag{4.1.4}$$

with the differential equation (4.1.1).

4.2 An asymptotic fundamental system for $K\eta = \tau^l H\eta$

We start this section with the following important theorem from [12] (see Theorem 8.2.1, page 326). The results and assumptions of this theorem are used later in our particular case.

Theorem 4.2.1. *Suppose that $h_{n_0} = 1$, set $l = n - n_0$, and let $k \in \mathbb{N}$. Suppose that $k \geq max\{l, n_0 - 1\}$ if $n_0 > 0$. Suppose that*

α) $k_j \in L_{p'}(a,b)$ for $j = 0, ..., n-1-k$ and $k_{n-1-j} \in W_{p'}^{k-j}(a,b)$ for $j = 0, ..., min\{k-l, n-1\}$ if $n_0 = 0$,

β) $h_0, ..., h_{n_0-1} \in W_{p'}^k(a,b)$, $k_0, ..., k_{n_0-1} \in W_{p'}^{k-l}(a,b)$, and $k_{n-1-j} \in W_{p'}^{k-j}(a,b)$ for $j = 0, ..., l-1$ if $n_0 > 0$. Let $\{\pi_1, ..., \pi_{n_0}\} \subset W_{p'}^{k+n_0}(a,b)$ be a fundamental system of $H\eta = 0$.

For sufficiently large τ the differential equation $K\eta = \tau^l H\eta$ has a fundamental system $\{\eta_1(\tau), ..., \eta_n(\tau)\}$ with the following properties:

i) *There are functions $\pi_{\nu r} \in W_{p'}^{k+n_0-lr}(a,b)$, $1 \leq \nu \leq n_0$, $1 \leq r \leq [\frac{k}{l}]$ such that*

$$\eta_\nu^{(\mu)}(.,\tau) = \pi_\nu^{(\mu)} + \sum_{r=1}^{[\frac{k}{l}]} \tau^{-lr}\pi_{\nu r}^{(\mu)} + \{o(\tau^{-k})\}_\infty, \quad \nu = 1, ..., n_0; \mu = 0, ..., n_0 - 1, \tag{4.2.1}$$

$$\eta_\nu^{(\mu)}(.,\tau) = \pi_\nu^{(\mu)} + \sum_{r=1}^{[\frac{k-\mu+n_0-1}{l}]} \tau^{-lr}\pi_{\nu r}^{(\mu)} + \{o(\tau^{-k+\mu-n_0+1})\}_\infty,$$

$$\nu = 1,...,n_0; \mu = n_0,...,n-1, \qquad (4.2.2)$$

ii) *Set* $\tilde{k} := min\{k, k+1-n_0\}$. *Let* $\omega_j = \exp\{\frac{2\pi i(j-1)}{l}\}$, $j = 1,...,l$. *There are functions* $\varphi_r \in W_{p'}^{k+1-r}(a,b)$, $r = 0,...,\tilde{k}$, *such that* φ_0 *is the only solution of the initial value problem*

$$\varphi_0' - \frac{1}{l}(h_{n_0-1} - k_{n-1})\varphi_0 = 0, \quad \varphi_0(a) = 1, \qquad (4.2.3)$$

and

$$\eta_\nu^{(\mu)}(x,\tau) = \left[\frac{d^\mu}{dx^\mu}\right]\{\sum_{r=0}^{\tilde{k}}(\tau\omega_{\nu-n_0})^{-r}\varphi_r(x)e^{\tau\omega_{\nu-n_0}(x-a)}\}$$
$$+ \{o(\tau^{-\tilde{k}+\mu})\}_\infty e^{\tau\omega_{\nu-n_0}(x-a)}, \quad \nu = n_0+1,...,n; \mu = 0,...,n-1, \qquad (4.2.4)$$

where $[\frac{d^\mu}{dx^\mu}]$ *means that we omit those terms of the Leibniz expansion which contain a derivative* $\varphi_r^{(j)}$ *with* $j > \tilde{k} - r$.

The proof of the above theorem is not given out here. We give the general assumptions required to carry on with the section that follows. Most of the assumptions are part of the assumptions given in the proof of the above theorem.

We shall denote the i-th unit vectors in $\mathbb{C}^n$, $\mathbb{C}^{n_0}$, $\mathbb{C}^l$ by e_i, ϵ_i, ε_i. For $i \in \mathbb{Z}\setminus\{1,...,n\}$ or $i \in \mathbb{Z}\setminus\{1,...,n_0\}$ or $i \in \mathbb{Z}\setminus\{1,...,l\}$ we set $e_i := 0$, $\epsilon_i := 0$, $\varepsilon_i := 0$, respectively. We set

$$a_1^\top(\tau) := \tau^l(h_0,...,h_{n_0-1}) - (k_0,...,k_{n_0-1}) =: \tau^l a_{11}^\top + a_{12}^\top, \qquad (4.2.5)$$

where

$$a_{11}^\top = (h_0,...,h_{n_0-1}), \quad a_{12}^\top = -(k_0,...,k_{n_0-1}).$$

Also set

$$a_2^\top := -(k_{n_0}, ..., k_{n-1}), \tag{4.2.6}$$

$$J_r := \begin{pmatrix} 0 & 1 & & & & \\ & 0 & . & & 0 & \\ & & . & . & & \\ & & & . & . & \\ & 0 & & & . & 1 \\ & & & & & 0 \end{pmatrix} \in M_r(\mathbb{C}), \tag{4.2.7}$$

$$\varepsilon^\top := \sum_{i=1}^{l} \varepsilon_i^\top = (1, ..., 1) \in \mathbb{C}^l, \tag{4.2.8}$$

$$\Omega_l := diag(\omega_1, ..., \omega_l), \tag{4.2.9}$$

$$\Xi_r(\tau) = diag(1, \tau, ..., \tau^{r-1}) \in M_r(\mathbb{C}) \tag{4.2.10}$$

and

$$V := \sum_{i=1}^{l} \varepsilon_i \varepsilon^\top \Omega_l^{i-1} = \begin{pmatrix} 1 & . & . & . & 1 \\ \omega_1 & . & . & . & \omega_l \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ \omega_1^{l-1} & . & . & . & \omega_l^{l-1} \end{pmatrix}. \tag{4.2.11}$$

When observing that

$$\varepsilon^\top \Omega_l^j \varepsilon = \sum_{i=1}^{l} \omega_i^j = \begin{cases} l & if \quad j = 0 mod(l), \\ 0 & if \quad j \neq 0 mod(l), \end{cases} \tag{4.2.12}$$

we obtain that V is invertible with

$$V^{-1} = \frac{1}{l} \sum_{i=1}^{l} \Omega_l^{1-i} \varepsilon \varepsilon_i^\top. \tag{4.2.13}$$

Also,

$$\varepsilon_i^\top V = \varepsilon^\top \Omega_l^{i-1} \quad i = 1, ..., l \tag{4.2.14}$$

and

$$(J_l + \varepsilon_l \varepsilon_1^\top)V = V\Omega_l. \tag{4.2.15}$$

We set $Q_{ij}^{[r]} := 0$ for $r < 0$ and consider the following equations:

$$Q_{11}^{[0]\prime} - (J_{n_0} - \epsilon_{n_0} a_{11}^\top) Q_{11}^{[0]} = 0, \quad Q_{11}^{[0]}(a) = I_{n_0}, \tag{4.2.16}$$

$$Q_{22}^{[0]\prime} - \frac{1}{l}(h_{n_0-1} - k_{n-1}) Q_{22}^{[0]} = 0, \quad Q_{22}^{[0]}(a) = I, \tag{4.2.17}$$

$$Q_{12}^{[0]} = 0, \quad Q_{21}^{[0]} = 0, \tag{4.2.18}$$

$$\begin{aligned} Q_{11}^{[r]\prime} - (J_{n_0} - \epsilon_{n_0} a_{11}^\top) Q_{11}^{[r]} &= (\epsilon_{n_0-1} - (h_{n_0-1} - k_{n-1})\epsilon_{n_0})\varepsilon^\top \Omega_l^{-1} Q_{21}^{[r]} \\ &+ \sum_{j=1}^{l} k_{n-1-j} \epsilon_{n_0} \varepsilon^\top \Omega_l^{-1-j} Q_{21}^{[r-j]} - \epsilon_{n_0} a_{12}^\top Q_{11}^{[r-l]}, \quad r = 1, ..., k, \end{aligned} \tag{4.2.19}$$

$$\begin{aligned} Q_{21}^{[r]} &= \Omega_l^{-1} Q_{21}^{[r-1]\prime} - \frac{1}{l}\varepsilon a_{11}^\top Q_{11}^{[r-1]} - \frac{1}{l}(h_{n_0-1} - k_{n-1})\varepsilon\varepsilon^\top \Omega_l^{-1} Q_{21}^{[r-1]} \\ &+ \frac{1}{l}\sum_{j=1}^{l} k_{n-1-j} \varepsilon\varepsilon^\top \Omega_l^{-1-j} Q_{21}^{[r-1-j]} - \frac{1}{l}\varepsilon a_{12}^\top Q_{11}^{[r-l-1]}, \quad r = 1, ..., k, \end{aligned} \tag{4.2.20}$$

$$\begin{aligned} Q_{12}^{[r]} &= -Q_{12}^{[r-1]\prime} \Omega_l^{-1} + (J_{n_0} - \epsilon_{n_0} a_{11}^\top) Q_{12}^{[r-1]} \Omega_l^{-1} \\ &+ (\epsilon_{n_0-1} - (h_{n_0-1} - k_{n-1})\epsilon_{n_0})\varepsilon^\top \Omega_l^{-1} Q_{22}^{[r-1]} \Omega_l^{-1} \\ &+ \sum_{j=1}^{l} k_{n-j-1} \epsilon_{n_0} \varepsilon^\top \Omega_l^{-1-j} \Omega_{22}^{[r-1-j]} \Omega_l^{-1} - \epsilon_{n_0} a_{12}^\top Q_{12}^{[r-1-l]} \Omega_l^{-1}, \quad r = 1, ..., k, \end{aligned} \tag{4.2.21}$$

$$\Omega_l Q_{22}^{[r]} - Q_{22}^{[r]}\Omega_l = Q_{22}^{[r-1]\prime} - \frac{1}{l}\Omega_l \varepsilon a_{11}^\top Q_{12}^{[r-1]} - \frac{1}{l}(h_{n_0-1} - k_{n-1})\Omega_l \varepsilon\varepsilon^\top \Omega_l^{-1} Q_{22}^{[r-1]}$$
$$+ \frac{1}{l}\sum_{j=1}^{l} k_{n-1-j}\Omega_l \varepsilon\varepsilon^\top \Omega_l^{-1-j} Q_{22}^{[r-1-j]} - \frac{1}{l}\Omega_l \varepsilon a_{12}^\top Q_{12}^{[r-1-l]}, \quad r = 1, ..., k \tag{4.2.22}$$

$$0 = \varepsilon_\nu^\top \{Q_{22}^{[k]\prime} - \frac{1}{l}\Omega_l \varepsilon a_{11}^\top Q_{12}^{[k]} - \frac{1}{l}(h_{n_0-1} - k_{n-1})\Omega_l \varepsilon\varepsilon^\top \Omega_l^{-1} Q_{22}^{[k]}$$
$$+ \frac{1}{l}\sum_{j=1}^{l} k_{n-1-j}\Omega_l \varepsilon\varepsilon^\top \Omega_l^{-1-j} Q_{22}^{[k-j]} - \frac{1}{l}\Omega_l \varepsilon a_{12}^\top Q_{12}^{[k-l]}\}\varepsilon_\nu, \quad \nu = 1, ..., l. \tag{4.2.23}$$

The above equations are used in the calculation of the functions φ_r in the subsequent section which are used in the calculation of the asymptotic fundamental system. From equation (4.2.22) for $r = 1$, we can conclude that the diagonal elements of the matrix $Q_{22}^{[0]}$ satisfy the differential equation

$$\eta' - \frac{1}{l}(h_{n_0} - k_{n-1})\eta = 0.$$

We also observe that the diagonal elements of $\Omega_l Q_{22}^{[1]} - Q_{22}^{[1]}\Omega_l$ are zero, that is $Q_{12}^{[0]} = 0$ by (4.2.18), and the diagonal elements of $\Omega_l \varepsilon\varepsilon^\top \Omega_l^{-1}$ have the value 1.

4.3 Asymptotics of eigenvalues of self-adjoint second order differential operators

In this section we now consider the differential equation in (1.3.1),

$$-y'' + gy' + hy = \lambda y.$$

4.3.1 The asymptotic fundamental system of $-y'' + gy' + hy = \lambda y$

For this case, we establish an eigenvalue asymptotics which is very precise. As before we set $\lambda = \mu^2$. The differential equation (1.3.1) is a special case of (4.1.1) with $n = 2$, $n_0 = 0$, $l = 2$ and

$$Ky = y'' - gy' - hy$$

and

$$Hy = y$$

It can be written as $Ky = -\lambda Hy$. Now, by matching the parameters we get that, $-\lambda = \tau^2$. Since $\lambda = \mu^2$, therefore we put $\tau = i\mu$. From equation (4.2.4) we have that (1.3.1) has an asymptotic fundamental system $\{y_1, y_2\}$ of the form

$$\begin{aligned}
y_\nu^{(j)}(x,\mu) &= \eta_\nu^{(j)}(x,\tau) \\
&= \eta_\nu^{(j)}(x,i\mu) \\
&= \left[\frac{d^j}{dx^j}\right]\left(\sum_{r=0}^{\tilde{k}}(i\mu\omega_{\nu-n_0})^{-r}\varphi_r(x)e^{\omega_{\nu-n_0}i\mu(x-a)}\right) \\
&+ \{o(\mu^{-\tilde{k}+j})\}e^{\omega_{\nu-n_0}i\mu(x-a)} \\
&= \left[\frac{d^j}{dx^j}\right]\left(\sum_{r=0}^{2}(i\mu\omega_\nu)^{-r}\varphi_r(x)e^{\omega_\nu i\mu x}\right) + (o(\mu^{-2+j}))e^{\omega_\nu i\mu x} \\
&= \delta_{\nu,j}(x,i\mu)e^{\omega_\nu i\mu x}
\end{aligned} \tag{4.3.1}$$

where

$$\delta_{\nu,j}(x,i\mu) = \left[\frac{d^j}{dx^j}\right]\left(\sum_{r=0}^{2}(i\mu\omega_\nu)^{-r}\varphi_r(x)e^{\omega_\nu i\mu x}\right)e^{-\omega_\nu i\mu x} + o(\mu^{-2+j}), \quad j = 0,1 \tag{4.3.2}$$

where $\omega_\nu = e^{\frac{2\pi i(\nu-1)}{2}} = e^{\pi i(\nu-1)}, \quad \nu = 1, 2.$

We will calculate φ_0, φ_1 and φ_2 and compare with the literature. Finally, we will use computer algebra to generate a code which explicitly allows automatic generation of the functions φ_1,φ_2 and the corresponding asymptotic expansions as in Theorem 1.2.1.

Observe from (4.1.2) and (4.1.3) that $n_0 = 0$ and $l = 2$, hence $n = 2$. From equation (8.2.45) in [12], we know that

$$\varphi_r = \varepsilon_1^\top V Q^{[r]} \varepsilon_1, \tag{4.3.3}$$

where ε_ν is the νth unit vector in $\mathbb{C}^2$, $Q^{[r]} := Q_{22}^{[r]}$ are 2×2 matrices and V, as a special case of equation (4.2.11), is

$$V = \begin{pmatrix} 1 & 1 \\ \omega_1 & \omega_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

$\omega_j = \exp\{\frac{2\pi i(j-1)}{2}\} = \exp\{\pi i(j-1)\} = (-1)^{(j-1)}, j = 1, 2.$

By assumption we know that $h_{n_0} > 0, n_0 \geq 0$. In our case $n_0 = 0$, $l = 2$ and $n = 2$, we therefore have that $h_{n_0-1} = 0$ and from equation (4.2.17) we have that

$$Q^{[0]\prime} + \frac{1}{2}k_1 Q^{[0]} = 0, \quad Q^{[0]}(0) = I. \tag{4.3.4}$$

For $r = 1$ in equation (4.2.22) we have that

$$\Omega_2 Q^{[1]} - Q^{[1]}\Omega_2 = Q^{[0]\prime} + \frac{1}{2}k_1 \Omega_2 \varepsilon\varepsilon^\top \Omega_2^{-1} Q^{[0]}. \tag{4.3.5}$$

From equation (4.2.17) we know that $Q_{12}^{[0]} = 0$ and from equation (4.2.5), $a_{11}^\top = (h_0, \ldots, h_{n_0-1}) = 0$ since $h_{n_0-1} = 0$, hence the disappearance of the second term on the right hand side. By assumption we know that $Q_{ij}^{[r]} = 0$ for $r < 0$, hence the disappearance of the last two terms on the right hand side.

For $r = 2$ in equation (4.2.22) we have that

$$\Omega_2 Q^{[2]} - Q^{[2]}\Omega_2 = Q^{[1]'} + \frac{1}{2}k_1\Omega_2\varepsilon\varepsilon^\top\Omega_2^{-1}Q^{[1]} + \frac{1}{2}k_0\Omega_2\varepsilon\varepsilon^\top\Omega_2^{-2}Q^{[0]}, \tag{4.3.6}$$

The second term on the right hand side disappears using the same argument above about $a_{11}^\top$. From equation (4.2.5) we have $a_{12}^\top = (k_0, \ldots, k_{n_0-1}) = 0$ since $k_{n_0-1} = 0$, hence the disappearance of the last two terms on the right hand side.

For $k = 2$ in equation (4.2.23). Using the same arguments as above for equations (4.3.5) and (4.3.6), we therefore have that

$$\begin{aligned} 0 &= \varepsilon_\nu^\top(Q^{[2]'} + \frac{1}{2}k_1\Omega_2\varepsilon\varepsilon^\top\Omega_2^{-1}Q^{[2]} + \frac{1}{2}\sum_{j=1}^{1}k_{1-j}\Omega_2\varepsilon\varepsilon^\top\Omega_2^{-1-j}Q^{[2-j]})\varepsilon_\nu \\ &= \varepsilon_\nu^\top(Q^{[2]'} + \frac{1}{2}k_1\Omega_2\varepsilon\varepsilon^\top\Omega_2^{-1}Q^{[2]} + \frac{1}{2}k_0\Omega_2\varepsilon\varepsilon^\top\Omega_2^{-2}Q^{[1]})\varepsilon_\nu, \quad \nu = 1, 2, \end{aligned} \tag{4.3.7}$$

In equations (4.3.4) to (4.3.7), $\Omega_2 = \mathrm{diag}(\omega_1, \omega_2) = \mathrm{diag}(1, -1)$, $k_0 = -h$ and $k_1 = -g$. Set

$$Q^{[0]} = \begin{pmatrix} q_{11}^0 & q_{12}^0 \\ q_{21}^0 & q_{22}^0 \end{pmatrix}.$$

From (4.3.4) we have that

$$\begin{pmatrix} q_{11}^{0'} - \frac{1}{2}gq_{11}^0 & q_{12}^{0'} - \frac{1}{2}gq_{12}^0 \\ q_{21}^{0'} - \frac{1}{2}gq_{21}^0 & q_{22}^{0'} - \frac{1}{2}gq_{22}^0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and thus

$$q_{11}^{0'} - \frac{1}{2}gq_{11}^0 = 0.$$

It is a first order linear differential equation in q_{11}^0 with integrating factor, $e^{-\frac{1}{2}G}$, where

$$G(x) = \int_0^x g(t)dt. \tag{4.3.8}$$

We now have that

$$
\begin{aligned}
(q_{11}^0 e^{-\frac{1}{2}G})' &= 0 \\
q_{11}^0 &= K_1 e^{\frac{1}{2}G}.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
q_{12}^0 &= K_2 e^{\frac{1}{2}G}, \\
q_{21}^0 &= K_3 e^{\frac{1}{2}G}
\end{aligned}
$$

and

$$q_{22}^0 = K_4 e^{\frac{1}{2}G},$$

where $K_j, j = 1, 2, 3, 4$ are constants. From (4.3.4) we have that

$$\begin{pmatrix} K_1 e^{\frac{1}{2}G(0)} & K_2 e^{\frac{1}{2}G(0)} \\ K_3 e^{\frac{1}{2}G(0)} & K_4 e^{\frac{1}{2}G(0)} \end{pmatrix} = \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$K_1 = K_4 = 1$ and $K_2 = K_3 = 0$. Thus

$$Q^{[0]} = \begin{pmatrix} e^{\frac{1}{2}G} & 0 \\ 0 & e^{\frac{1}{2}G} \end{pmatrix}.$$

Now, we set

$$Q^{[1]} = \begin{pmatrix} q_{11}^1 & q_{12}^1 \\ q_{21}^1 & q_{22}^1 \end{pmatrix},$$

$$\Omega_2 Q^{[1]} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} q_{11}^1 & q_{12}^1 \\ q_{21}^1 & q_{22}^1 \end{pmatrix} = \begin{pmatrix} q_{11}^1 & q_{12}^1 \\ -q_{21}^1 & -q_{22}^1 \end{pmatrix}$$

and

$$Q^{[1]}\Omega_2 = \begin{pmatrix} q_{11}^1 & q_{12}^1 \\ q_{21}^1 & q_{22}^1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} q_{11}^1 & -q_{12}^1 \\ q_{21}^1 & -q_{22}^1 \end{pmatrix}.$$

Thus

$$\Omega_2 Q^{[1]} - Q^{[1]}\Omega_2 = \begin{pmatrix} q_{11}^1 & q_{12}^1 \\ -q_{21}^1 & -q_{22}^1 \end{pmatrix} - \begin{pmatrix} q_{11}^1 & -q_{12}^1 \\ q_{21}^1 & -q_{22}^1 \end{pmatrix} = \begin{pmatrix} 0 & 2q_{12}^1 \\ -2q_{21}^1 & 0 \end{pmatrix}. \tag{4.3.9}$$

It can be checked that

$$\varepsilon\varepsilon^\top = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \Omega_2^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \Omega_2^{-2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and thus we have

$$\varepsilon\varepsilon^\top\Omega_2^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix},$$

$$\Omega_2\varepsilon\varepsilon^\top\Omega_2^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

and

$$\begin{aligned} \frac{1}{2}g\Omega_2\varepsilon\varepsilon^\top\Omega_2^{-1}Q^{[0]} &= \frac{1}{2}g\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} e^{\frac{1}{2}G} & 0 \\ 0 & e^{\frac{1}{2}G} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}ge^{\frac{1}{2}G} & -\frac{1}{2}ge^{\frac{1}{2}G} \\ -\frac{1}{2}ge^{\frac{1}{2}G} & \frac{1}{2}ge^{\frac{1}{2}G} \end{pmatrix}. \end{aligned}$$

From (4.3.5) we know that $\Omega_2 Q^{[1]} - Q^{[1]}\Omega_2 = Q^{[0]\prime} - \frac{1}{2}g\Omega_2\varepsilon\varepsilon^\top\Omega_2^{-1}Q^{[0]}$, where

$$\begin{aligned} Q^{[0]\prime} - \frac{1}{2}g\Omega_2\varepsilon\varepsilon^\top\Omega_2^{-1}Q^{[0]} &= \begin{pmatrix} \frac{1}{2}ge^{\frac{1}{2}G} & 0 \\ 0 & \frac{1}{2}ge^{\frac{1}{2}G} \end{pmatrix} - \begin{pmatrix} \frac{1}{2}ge^{\frac{1}{2}G} & -\frac{1}{2}ge^{\frac{1}{2}G} \\ -\frac{1}{2}ge^{\frac{1}{2}G} & \frac{1}{2}ge^{\frac{1}{2}G} \end{pmatrix} \\ &= \begin{pmatrix} 0 & \frac{1}{2}ge^{\frac{1}{2}G} \\ \frac{1}{2}ge^{\frac{1}{2}G} & 0 \end{pmatrix}. \end{aligned} \tag{4.3.10}$$

It results from (4.3.9) and (4.3.10) that $q_{12}^1 = \frac{1}{4}ge^{\frac{1}{2}G}$, $q_{21}^1 = -\frac{1}{4}ge^{\frac{1}{2}G}$ and

$$Q^{[1]} = \begin{pmatrix} q_{11}^1 & \frac{1}{4}ge^{\frac{1}{2}G} \\ -\frac{1}{4}ge^{\frac{1}{2}G} & q_{22}^1 \end{pmatrix}. \tag{4.3.11}$$

Set

$$Q^{[2]} = \begin{pmatrix} q_{11}^2 & q_{12}^2 \\ q_{21}^2 & q_{22}^2 \end{pmatrix}.$$

It follows from (4.3.6) that $\Omega_2 Q^{[2]} - Q^{[2]}\Omega_2 = Q^{[1]'} + \frac{1}{2}k_1\Omega_2\varepsilon\varepsilon^\top\Omega_2^{-1}Q^{[1]} + \frac{1}{2}k_0\Omega_2\varepsilon\varepsilon^\top\Omega_2^{-2}Q^{[0]}$ and similarly as in (4.3.9) we have that

$$\Omega_2 Q^{[2]} - Q^{[2]}\Omega_2 = \begin{pmatrix} 0 & 2q_{12}^2 \\ -2q_{21}^2 & 0 \end{pmatrix}. \tag{4.3.12}$$

Hence from (4.3.6) we have that

$$\begin{aligned}
&Q^{[1]'} - \frac{1}{2}g\Omega_2\varepsilon\varepsilon^\top\Omega_2^{-1}Q^{[1]} - \frac{1}{2}h\Omega_2\varepsilon\varepsilon^\top\Omega_2^{-2}Q^{[0]} \\
&= \begin{pmatrix} q_{11}^{1'} & \frac{1}{4}(ge^{\frac{1}{2}G})' \\ -\frac{1}{4}(ge^{\frac{1}{2}G})' & q_{22}^{1'} \end{pmatrix} \\
&+ \begin{pmatrix} -\frac{1}{2}gq_{11}^1 - \frac{1}{8}g^2e^{\frac{1}{2}G} & \frac{1}{2}gq_{22}^1 - \frac{1}{8}g^2e^{\frac{1}{2}G} \\ \frac{1}{2}gq_{11}^1 + \frac{1}{8}g^2e^{\frac{1}{2}G} & -\frac{1}{2}gq_{22}^1 + \frac{1}{8}g^2e^{\frac{1}{2}G} \end{pmatrix} \\
&+ \begin{pmatrix} -\frac{1}{2}he^{\frac{1}{2}G} & -\frac{1}{2}he^{\frac{1}{2}G} \\ \frac{1}{2}he^{\frac{1}{2}G} & \frac{1}{2}he^{\frac{1}{2}G} \end{pmatrix} \\
&= \begin{pmatrix} q_{11}^{1'} - \frac{1}{2}gq_{11}^1 - \frac{1}{8}(g^2+4h)e^{\frac{1}{2}G} & \frac{1}{4}(ge^{-\frac{1}{2}G})' + \frac{1}{2}gq_{22}^1 - \frac{1}{8}(g^2+4h)e^{\frac{1}{2}G} \\ -\frac{1}{4}(ge^{\frac{1}{2}G})' + \frac{1}{2}gq_{11}^1 + \frac{1}{8}(g^2+4h)e^{\frac{1}{2}G} & q_{22}^{1'} - \frac{1}{2}gq_{22}^1 + \frac{1}{8}(g^2+4h)e^{\frac{1}{2}G} \end{pmatrix}.
\end{aligned} \tag{4.3.13}$$

By comparing (4.3.12) and (4.3.13) we get that

$$q_{11}^{1'} - \frac{1}{2}gq_{11}^1 - \frac{1}{8}(g^2+4h)e^{\frac{1}{2}G} = 0, \tag{4.3.14}$$

$$\frac{1}{4}(ge^{\frac{1}{2}G})' + \frac{1}{2}gq_{22}^{1} - \frac{1}{8}(g^2+4h)e^{\frac{1}{2}G} = 2q_{12}^{2}, \tag{4.3.15}$$

$$-\frac{1}{4}(ge^{\frac{1}{2}G})' + \frac{1}{2}gq_{11}^{1} + \frac{1}{8}(g^2+4h)e^{\frac{1}{2}G} = -2q_{21}^{2} \tag{4.3.16}$$

and

$$q_{22}^{1'} - \frac{1}{2}gq_{22}^{1} + \frac{1}{8}(g^2+4h)e^{\frac{1}{2}G} = 0. \tag{4.3.17}$$

Equations (4.3.14) and (4.3.17) are first order linear differential equations in q_{11}^1 and q_{22}^1 respectively and can be written as

$$q_{11}^{1'} - \frac{1}{2}gq_{11}^{1} = \frac{1}{8}(g^2+4h)e^{\frac{1}{2}G}$$

and

$$q_{22}^{1'} - \frac{1}{2}gq_{22}^{2} = -\frac{1}{8}(g^2+4h)e^{\frac{1}{2}G}.$$

By multiplying each term of (4.3.14) by the integrating factor, $e^{-\frac{1}{2}\int gdt} = e^{-\frac{1}{2}G}$, we have

$$\begin{aligned}
(q_{11}^{1}e^{-\frac{1}{2}G})' &= \frac{1}{8}(g^2+4h)\\
q_{11}^{1}e^{-\frac{1}{2}G} &= \frac{1}{8}\int(g^2+4h)dx\\
q_{11}^{1} &= \frac{1}{8}e^{\frac{1}{2}G}\int(g^2+4h)dx\\
q_{11}^{1} &= \frac{1}{8}Ie^{\frac{1}{2}G},
\end{aligned}$$

where

$$I(x) = \int_0^x (g^2(t)+4h(t))dt. \tag{4.3.18}$$

Similarly, for equation (4.3.17) we have

$$\begin{aligned}
(q^1_{22}e^{-\frac{1}{2}G})' &= -\frac{1}{8}(g^2+4h)\\
q^1_{22}e^{-\frac{1}{2}G} &= -\frac{1}{8}\int(g^2+4h)dx\\
q^1_{22} &= -\frac{1}{8}e^{\frac{1}{2}G}\int(g^2+4h)dx\\
q^1_{22} &= -\frac{1}{8}Ie^{\frac{1}{2}G}.
\end{aligned}$$

Thus substituting q^1_{22} in equation (4.3.15) we get that

$$\begin{aligned}
q^2_{12} &= \frac{1}{8}(ge^{\frac{1}{2}G})' + \frac{1}{4}gq^1_{22} - \frac{1}{16}(g^2+4h)e^{\frac{1}{2}G}\\
&= \frac{1}{8}(g'e^{\frac{1}{2}G} + \frac{1}{2}g^2e^{\frac{1}{2}G}) + \frac{1}{4}g(-\frac{1}{8}Ie^{\frac{1}{2}G}) - \frac{1}{16}(g^2+4h)e^{\frac{1}{2}G}\\
&= \frac{1}{8}g'e^{-\frac{1}{2}G} + \frac{1}{16}g^2e^{\frac{1}{2}G} - \frac{1}{32}gIe^{\frac{1}{2}G} - \frac{1}{16}g^2e^{\frac{1}{2}G} - \frac{1}{4}he^{\frac{1}{2}G}\\
&= \frac{1}{32}e^{\frac{1}{2}G}(4g' - gI - 8h)
\end{aligned}$$

and by substituting q^1_{11} in equation (4.3.16) we see that

$$\begin{aligned}
q^2_{21} &= \frac{1}{8}(ge^{\frac{1}{2}G})' - \frac{1}{4}gq^1_{11} - \frac{1}{16}(g^2+4h)e^{\frac{1}{2}G}\\
&= \frac{1}{8}(g'e^{\frac{1}{2}G} + \frac{1}{2}g^2e^{\frac{1}{2}G}) - \frac{1}{4}g(\frac{1}{8}Ie^{\frac{1}{2}G}) - \frac{1}{16}(g^2+4h)e^{\frac{1}{2}G}\\
&= \frac{1}{8}g'e^{\frac{1}{2}G} + \frac{1}{16}g^2e^{\frac{1}{2}G} - \frac{1}{32}gIe^{\frac{1}{2}G} - \frac{1}{16}g^2e^{\frac{1}{2}G} - \frac{1}{4}he^{\frac{1}{2}G}\\
&= \frac{1}{32}e^{\frac{1}{2}G}(4g' - gI - 8h)\\
&= q^2_{12}.
\end{aligned}$$

Hence

$$Q^{[1]} = \begin{pmatrix} \frac{1}{8}Ie^{\frac{1}{2}G} & \frac{1}{4}ge^{\frac{1}{2}G} \\ -\frac{1}{4}ge^{\frac{1}{2}G} & -\frac{1}{8}Ie^{\frac{1}{2}G} \end{pmatrix} \tag{4.3.19}$$

and

$$Q^{[2]} = \begin{pmatrix} q_{11}^2 & \frac{1}{32}Je^{\frac{1}{2}G} \\ \frac{1}{32}Je^{\frac{1}{2}G} & q_{22}^2 \end{pmatrix}, \tag{4.3.20}$$

where

$$J(x) = 4g'(x) - g(x)I(x) - 8h(x). \tag{4.3.21}$$

We now set

$$Q^{[3]} = \begin{pmatrix} q_{11}^3 & q_{12}^3 \\ q_{21}^3 & q_{22}^3 \end{pmatrix}. \tag{4.3.22}$$

For $r = 3$ in equation (4.2.22). Using the same arguments about $a_{11}^\top$ and $a_{12}^\top$ as in equations (4.3.5) and (4.3.6) we then have,

$$\Omega_2 Q^{[3]} - Q^{[3]}\Omega_2 = Q^{[2]'} + \frac{1}{2}k_1\Omega_2\varepsilon\varepsilon^\top\Omega_2^{-1}Q^{[2]} + \frac{1}{2}k_0\Omega_2\varepsilon\varepsilon^\top\Omega_2^{-2}Q^{[1]}. \tag{4.3.23}$$

Similarly as in equations (4.3.9), (4.3.12) and (4.3.13) we have

$$\Omega_2 Q^{[3]} - Q^{[3]}\Omega_2 = \begin{pmatrix} 0 & 2q_{12}^3 \\ -2q_{21}^3 & 0 \end{pmatrix}, \tag{4.3.24}$$

$$\begin{aligned} \frac{1}{2}g\Omega_2\varepsilon\varepsilon^\top\Omega_2^{-1}Q^{[2]} &= \begin{pmatrix} \frac{1}{2}g & -\frac{1}{2}g \\ -\frac{1}{2}g & \frac{1}{2}g \end{pmatrix}\begin{pmatrix} q_{11}^2 & q_{12}^2 \\ q_{21}^2 & q_{22}^2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}g(q_{11}^2 - q_{21}^2) & -\frac{1}{2}g(q_{22}^2 - q_{12}^2) \\ -\frac{1}{2}g(q_{11}^2 - q_{21}^2) & \frac{1}{2}g(q_{22}^2 - q_{12}^2) \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} \frac{1}{2}h\Omega_2\varepsilon\varepsilon^\top\Omega_2^{-2}Q^{[1]} &= \begin{pmatrix} \frac{1}{2}h & \frac{1}{2}h \\ -\frac{1}{2}h & -\frac{1}{2}h \end{pmatrix}\begin{pmatrix} \frac{1}{8}Ie^{\frac{1}{2}G} & \frac{1}{4}ge^{\frac{1}{2}G} \\ -\frac{1}{4}ge^{\frac{1}{2}G} & -\frac{1}{8}Ie^{\frac{1}{2}G} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{16}he^{\frac{1}{2}G}(I-2g) & -\frac{1}{16}he^{\frac{1}{2}G}(I-2g) \\ -\frac{1}{16}he^{\frac{1}{2}G}(I-2g) & \frac{1}{16}he^{\frac{1}{2}G}(I-2g) \end{pmatrix} \end{aligned}$$

and

$$
\begin{aligned}
&\Omega_2 Q^{[3]} - Q^{[3]}\Omega_2 \\
&= Q^{[2]'} - \frac{1}{2} g \Omega_2 \varepsilon\varepsilon^\top \Omega_2^{-1} Q^{[2]} - \frac{1}{2} h \Omega_2 \varepsilon\varepsilon^\top \Omega_2^{-2} Q^{[1]} \\
&= \begin{pmatrix} q_{11}^{2'} & q_{12}^{2'} \\ q_{21}^{2'} & q_{22}^{2'} \end{pmatrix} - \begin{pmatrix} \frac{1}{2} g(q_{11}^2 - q_{21}^2) & -\frac{1}{2} g(q_{22}^2 - q_{12}^2) \\ -\frac{1}{2} g(q_{11}^2 - q_{21}^2) & \frac{1}{2} g(q_{22}^2 - q_{12}^2) \end{pmatrix} \\
&- \begin{pmatrix} \frac{1}{16} h e^{\frac{1}{2}G}(I-2g) & -\frac{1}{16} h e^{\frac{1}{2}G}(I-2g) \\ -\frac{1}{16} h e^{\frac{1}{2}G}(I-2g) & \frac{1}{16} h e^{\frac{1}{2}G}(I-2g) \end{pmatrix} \\
&= \begin{pmatrix} q_{11}^{2'} - \frac{1}{2} g(q_{11}^2 - q_{21}^2) - \frac{1}{16} h e^{\frac{1}{2}G}(I-2g) & q_{12}^{2'} + \frac{1}{2} g(q_{22}^2 - q_{12}^2) + \frac{1}{16} h e^{\frac{1}{2}G}(I-2g) \\ q_{21}^{2'} + \frac{1}{2} g(q_{11}^2 - q_{21}^2) + \frac{1}{16} h e^{\frac{1}{2}G}(I-2g) & q_{22}^{2'} - \frac{1}{2} g(q_{22}^2 - q_{12}^2) - \frac{1}{16} h e^{\frac{1}{2}G}(I-2g) \end{pmatrix}.
\end{aligned}
\tag{4.3.25}
$$

By comparing the diagonal entries of (4.3.24) and (4.3.25) we get

$$
q_{11}^{2'} - \frac{1}{2} g(q_{11}^2 - q_{21}^2) - \frac{1}{16} h e^{\frac{1}{2}G}(I-2g) = 0 \tag{4.3.26}
$$

and

$$
q_{22}^{2'} - \frac{1}{2} g(q_{22}^2 - q_{12}^2) - \frac{1}{16} h e^{\frac{1}{2}G}(I-2g) = 0. \tag{4.3.27}
$$

Equation (4.3.26) is a linear first order differential equation in q_{11}^2 with integrating factor $e^{-\frac{1}{2}G}$ and can be written as,

$$
\begin{aligned}
q_{11}^{2'} - \frac{1}{2} g q_{11}^2 + \frac{1}{2} g\left(\frac{1}{32} J e^{\frac{1}{2}G}\right) - \frac{1}{16} h e^{\frac{1}{2}G}(I-2g) = 0 \\
q_{11}^{2'} - \frac{1}{2} g q_{11}^2 = \left(\frac{1}{16} h(I-2g) - \frac{1}{64} gJ\right) e^{\frac{1}{2}G}
\end{aligned}
$$

Multiplying each term with the integrating factor $e^{-\frac{1}{2}G}$, we get

$$
(q_{11}^2 e^{-\frac{1}{2}G})' = \frac{1}{64}(4h(I-2g) - gJ)
$$

$$
\begin{aligned}
q_{11}^2 e^{-\frac{1}{2}G} &= \frac{1}{64}\int (4h(I-2g) - gJ))dx \\
q_{11}^2 &= \frac{1}{64} K e^{\frac{1}{2}G}
\end{aligned}
$$

and similarly, since q_{21}^2 satisfies the same differential equation as q_{12}^2 we have that

$$
q_{22}^2 = \frac{1}{64} K e^{\frac{1}{2}G},
$$

where

$$
K(x) = \int_0^x (4h(t)(I(t) - 2g(t)) - g(t)J(t))dt. \tag{4.3.28}
$$

Thus

$$
Q^{[2]} = \begin{pmatrix} \frac{1}{64} K e^{\frac{1}{2}G} & \frac{1}{32} J e^{\frac{1}{2}G} \\ \frac{1}{32} J e^{\frac{1}{2}G} & \frac{1}{64} K e^{\frac{1}{2}G} \end{pmatrix}. \tag{4.3.29}
$$

We now compute the functions φ_0, φ_1 and φ_2. For $r = 0$ in equation (4.3.3) we have that

$$
\begin{aligned}
\varphi_0 &= \varepsilon_1^\top V Q^{[0]} \varepsilon_1 \\
&= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^{\frac{1}{2}G} & 0 \\ 0 & e^{\frac{1}{2}G} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= e^{\frac{1}{2}G}.
\end{aligned} \tag{4.3.30}
$$

For $r = 1$ in equation (4.3.3) we have that

$$
\begin{aligned}
\varphi_1 &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{8} I e^{\frac{1}{2}G} & \frac{1}{4} g e^{\frac{1}{2}G} \\ -\frac{1}{4} g e^{\frac{1}{2}G} & -\frac{1}{8} I e^{\frac{1}{2}G} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \frac{1}{8} e^{\frac{1}{2}G} (I - 2g).
\end{aligned} \tag{4.3.31}
$$

For $r = 2$ in equation (4.3.3) we have that

$$\begin{aligned}\varphi_2 &= \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}\frac{1}{64}Ke^{\frac{1}{2}G} & \frac{1}{32}Je^{\frac{1}{2}G}\\ \frac{1}{32}Je^{\frac{1}{2}G} & \frac{1}{64}Ke^{\frac{1}{2}G}\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}\\ &= \frac{1}{64}e^{\frac{1}{2}G}(2J+K).\end{aligned} \tag{4.3.32}$$

Thus from equation (4.3.2) we have

$$\begin{aligned}\delta_{\nu,j}(x,i\mu) = [\frac{d^j}{dx^j}](\varphi_0(x)e^{\omega_\nu i\mu x} + (i\mu\omega_\nu)^{-1}\varphi_1(x)e^{\omega_\nu i\mu x} + (i\mu\omega_\nu)^{-2}\varphi_2(x)e^{\omega_\nu i\mu x})e^{-\omega_\nu i\mu x}\\ + o(\mu^{-2+j}), \quad j = 0,1,\end{aligned} \tag{4.3.33}$$

where

$$\begin{aligned}\delta_{\nu,0} &= (\varphi_0 e^{\omega_\nu i\mu x} + (i\mu\omega_\nu)^{-1}\varphi_1 e^{\omega_\nu i\mu x} + (i\mu\omega_\nu)^{-2}\varphi_2 e^{\omega_\nu i\mu x})e^{-\omega_\nu i\mu x} + o(\mu^{-2})\\ &= (1 + \frac{1}{8}(i\mu\omega_\nu)^{-1}(I-2g) + \frac{1}{64}(i\mu\omega_\nu)^{-2}(2J+K))e^{\frac{1}{2}G} + o(\mu^{-2}), \quad \nu = 1,2\end{aligned} \tag{4.3.34}$$

and

$$\begin{aligned}\delta_{\nu,1} &= [\frac{d}{dx}](\varphi_0 e^{\omega_\nu i\mu x} + (i\mu\omega_\nu)^{-1}\varphi_1 e^{\omega_\nu i\mu x} + (i\mu\omega_\nu)^{-2}\varphi_2 e^{\omega_\nu i\mu x})e^{-\omega_\nu i\mu x} + o(\mu^{-1})\\ &= (i\mu\omega_\nu\varphi_0 e^{\omega_\nu i\mu x} + \varphi_0' e^{\omega_\nu i\mu x} + (i\mu\omega_\nu)^{-1}i\mu\omega_\nu\varphi_1 e^{\omega_\nu i\mu x} + (\mu\omega_\nu)^{-1}\varphi_1' e^{\omega_\nu \mu x}\\ &+ (i\mu\omega_\nu)^{-2}i\mu\omega_\nu\varphi_2 e^{\omega_\nu i\mu x} + (i\mu\omega_\nu)^{-2}\varphi_2' e^{\omega_\nu i\mu x})e^{-\omega_\nu i\mu x} + o(\mu^{-1})\\ &= i\mu\omega_\nu\varphi_0 + \varphi_0' + \varphi_1 + (i\mu\omega_\nu)^{-1}\varphi_1' + (i\mu\omega_\nu)^{-1}\varphi_2 + o(\mu^{-1})\\ &= i\mu\omega_\nu e^{\frac{1}{2}G} + \frac{1}{2}ge^{\frac{1}{2}G} + \frac{1}{8}(I-2g)e^{\frac{1}{2}G}\\ &+ \frac{1}{8}(i\mu\omega_\nu)^{-1}(\frac{1}{2}ge^{\frac{1}{2}G}(I-2g) + (I'-2g')e^{\frac{1}{2}G}) + \frac{1}{64}(i\mu\omega_\nu)^{-1}(2J+K)e^{\frac{1}{2}G}\\ &+ o(\mu^{-1})\\ &= (i\mu\omega_\nu + \frac{1}{4}g + \frac{1}{8}I + \frac{1}{64}(i\mu\omega_\nu)^{-1}(2J+K+4gI-8g^2+8I'-16g'))e^{\frac{1}{2}G}\end{aligned}$$

$$+ o(\mu^{-1}), \quad \nu = 1, 2. \tag{4.3.35}$$

Thus from (4.3.1) we have

$$\begin{aligned}
y_\nu &= \delta_{\nu,0} e^{\omega_\nu i\mu x} \\
&= (e^{\frac{1}{2}G} + \frac{1}{8}(i\mu\omega_\nu)^{-1}(I - 2g)e^{\frac{1}{2}G} + \frac{1}{64}(i\mu\omega_\nu)^{-2}(2J + K)e^{\frac{1}{2}G} + o(\mu^{-2}))e^{\omega_\nu i\mu x} \\
&= (1 + \frac{1}{8}(i\mu\omega_\nu)^{-1}(I - 2g) + \frac{1}{64}(i\mu\omega_\nu)^{-2}(2J + K))e^{\frac{1}{2}G}e^{\omega_\nu i\mu x} \\
&+ o(\mu^{-2})e^{\omega_\nu i\mu x}, \quad \nu = 1, 2.
\end{aligned} \tag{4.3.36}$$

In this section we have computed the asymptotic fundamental system $\{y_1, y_2\}$ of $-y'' + gy' + hy = \mu^2 y$ and this concludes the section. We now have the tools to calculate the asymptotic eigenvalues of the eigenvalue problems, $\cos\alpha y(\mu, 0) + \sin\alpha y'(\mu, 0)$, $\cos\alpha y(\mu, a) + \sin\alpha y'(\mu, a)$ and $y(\mu, 0) - y(\mu, a)$, $y'(\mu, 0) - y'(\mu, a)$ for general g, h.

4.3.2 Asymptotics of the eigenvalues for $B_1 y = \cos\alpha y(0, \mu) + \sin\alpha y'(0, \mu)$ and $B_2 y = \cos\beta y(a, \mu) + \sin\beta y'(a, \mu)$

Here we investigate the asymptotic eigenvalues of the eigenvalue problem (1.3.1) – (1.3.3) with $g, h \neq 0$. By using equation (4.3.33) and the set of boundary conditions $B_j y = 0, j = 1, 2$, we denote the entries of the characteristic matrix $M(\mu^2) = (B_i y_j(\mu^2))_{i,j}^2$ (the equivalence of equation (3.0.1)) by

$$\alpha_{j,k} = \gamma_{j,k} \exp(\varepsilon_{j,k}), \quad j, k = 1, 2,$$

where

$$\varepsilon_{1,k} = 0, \quad \varepsilon_{2,k} = \omega_k i\mu a,$$

$$\begin{aligned}\gamma_{1,k} &= \cos\alpha\delta_{k,0}(0,\mu) + \sin\alpha\delta_{k,1}(0,\mu)\\ \gamma_{2,k} &= \cos\beta\delta_{k,0}(a,\mu) + \sin\beta\delta_{k,1}(a,\mu)\end{aligned} \tag{4.3.37}$$

for the first set of the boundary conditions, that is the boundary condition in equation (1.3.2).

In this section we compute the asymptotic eigenvalues for the first set of the boundary conditions. The characteristic determinant of matrix M for general g, h is given by

$$\begin{aligned}D(\mu) &= \det(\gamma_{j,k}\exp(\varepsilon_{j,k}))_{j,k}^{2}\\ &= \det\begin{pmatrix}\gamma_{1,1} & \gamma_{1,2}\\ \gamma_{2,1}e^{i\mu a} & \gamma_{2,2}e^{-i\mu a}\end{pmatrix}\\ &= \gamma_{1,1}\gamma_{2,2}e^{-i\mu a} - \gamma_{2,1}\gamma_{1,2}e^{i\mu a}\\ &:= \psi_1 e^{-i\mu a} - \psi_2 e^{i\mu a}.\end{aligned} \tag{4.3.38}$$

Let D, as defined above be the characteristic function of the problem (1.3.1)-(1.3.2) with respect to $y_j, j = 1, 2$ with $y^{(j)} = \delta_{\nu,j}$ for $j = 0, 1$. The asymptotic distribution of the eigenvalues for $g, h = 0$ are known already as estimated in chapter 3. We denote by ϕ_0 the characteristic determinant of the eigenvalue problem for $g, h = 0$ as discussed in chapter 3. The functions g and h have an influence on terms of the lower order in D. D and ϕ_0 are asymptotically close to each other. The eigenvalue problem for general functions g and h has the same leading terms as the eigenvalue problem for $g, h = 0$. The eigenvalue asymptotics along the positive real axis will be determined since the problem is symmetric. From Theorem 3.1.1 we know that the zeros of ϕ_0 satisfy the asymptotics $\mu_k = \frac{\pi}{a}k + \rho_0 + o(1)$ as $k \to \infty$. The asymptotics can be improved by letting,

$$\mu_k = \frac{\pi}{a}k + \rho(k), \quad \rho(k) = \sum_{m=0}^{n} \rho_k k^{-m} + o(k^{-n}), \quad k = 1, 2, \ldots \tag{4.3.39}$$

We then find the terms ρ_0, ρ_1 and ρ_2 in the expansion of $\rho(k)$. The exponential function in equation (4.3.38) can be expanded using the second order Taylor expansion of the function $x \mapsto e^x$ in the neighborhood of 0 for large k, and we get

$$\begin{aligned} e^{i\mu_k a} =& e^{i\pi k} e^{i\rho(k)a} = e^{i\pi k} e^{i\rho_0 a} \exp(ia(\frac{\rho_1}{k} + \frac{\rho_2}{k^2} + o(k^{-2}))) \\ =& (-1)^k e^{i\rho_0 a}(1 + ia\frac{\rho_1}{k} + (ia\rho_2 - \frac{1}{2}\rho_1^2 a^2)\frac{1}{k^2} + o(k^{-2})), \end{aligned} \tag{4.3.40}$$

and

$$\begin{aligned} e^{-i\mu_k a} =& e^{-i\pi k} e^{-i\rho(k)a} = e^{-i\pi k} e^{-i\rho_0 a} \exp(-ia(\frac{\rho_1}{k} + \frac{\rho_2}{k^2} + o(k^{-2}))) \\ =& (-1)^k e^{-i\rho_0 a}(1 - ia\frac{\rho_1}{k} - (\frac{1}{2}\rho_1^2 a^2 + ia\rho_2)\frac{1}{k^2} + o(k^{-2})). \end{aligned} \tag{4.3.41}$$

From (4.3.38) we can write $D(\mu_k) = 0$ as

$$\psi_1 e^{-i\rho(k)a} - \psi_2 e^{i\rho(k)a} = 0. \tag{4.3.42}$$

Where $\psi_1 = \gamma_{1,1}\gamma_{2,2}$ and $\psi_2 = \gamma_{1,2}\gamma_{2,1}$. From (4.3.34), (4.3.35) and using (4.3.8), (4.3.18), (4.3.21) and (4.3.28) we get the following

$$\begin{aligned} \delta_{1,0}(0, \mu_k) &= 1 - \frac{1}{4}(i\mu_k)^{-1}g(0) + \frac{1}{32}(i\mu_k)^{-2}J(0) + o(\mu_k^{-2}), \\ \delta_{2,0}(0, \mu_k) &= 1 - \frac{1}{4}(-i\mu_k)^{-1}g(0) + \frac{1}{32}(-i\mu_k)^{-2}J(0) + o(\mu_k^{-2}), \\ \delta_{1,0}(a, \mu_k) &= (1 + \frac{1}{8}(i\mu_k)^{-1}(I(a) - 2g(a)) + \frac{1}{64}(i\mu_k)^{-2}(K(a) + 2J(a)))e^{\frac{1}{2}G(a)} \\ &\quad + o(\mu_k^{-2}), \\ \delta_{2,0}(a, \mu_k) &= (1 + \frac{1}{8}(-i\mu_k)^{-1}(I(a) - 2g(a)) + \frac{1}{64}(-i\mu_k)^{-2}(K(a) + 2J(a)))e^{\frac{1}{2}G(a)} \\ &\quad + o(\mu_k^{-2}), \end{aligned}$$

$$\begin{aligned}
\delta_{1,1}(0,\mu_k) &= i\mu_k + \frac{1}{4}g(0) + \frac{1}{64}(i\mu_k)^{-1}(2J(0) - 8g^2(0) + 8I'(0) - 16g'(0)) + o(\mu_k^{-1}),\\
\delta_{2,1}(0,\mu_k) &= -i\mu_k + \frac{1}{4}g(0) + \frac{1}{64}(-i\mu_k)^{-1}(2J(0) - 8g^2(0) + 8I'(0) - 16g'(0))\\
&\quad + o(\mu_k^{-1}),\\
\delta_{1,1}(a,\mu_k) &= (i\mu_k + \frac{1}{4}g(a) + \frac{1}{8}I(a)\\
&\quad + \frac{1}{64}(i\mu_k)^{-1}(2J(a) + K(a) + 4g(a)I(a) - 8g^2(a) + 8I'(a)\\
&\quad - 16g'(a)))e^{\frac{1}{2}G(a)} + o(\mu_k^{-1}),\\
\delta_{2,1}(a,\mu_k) &= (-i\mu_k + \frac{1}{4}g(a) + \frac{1}{8}I(a)\\
&\quad + \frac{1}{64}(-i\mu_k)^{-1}(2J(a) + K(a) + 4g(a)I(a) - 8g^2(a) + 8I'(a)\\
&\quad - 16g'(a)))e^{\frac{1}{2}G(a)} + o(\mu_k^{-1}).
\end{aligned}$$

We now apply the second order Taylor expansion of the function $x \mapsto \frac{1}{1+x}$ in the neighborhood of 0 to $\frac{1}{\mu_k}$ where k is large, and get that

$$\frac{1}{\mu_k} = \frac{a}{k\pi}\left(\frac{1}{1+\frac{a\rho(k)}{k\pi}}\right) = \frac{a}{k\pi}\left(1+\frac{a\rho(k)}{k\pi}\right)^{-1} = \frac{a}{k\pi} - \frac{a^2\rho_0}{k^2\pi^2} + o(k^{-2}). \tag{4.3.43}$$

For $k = 1, 2$ in equation (4.3.37) we have the following

$$\begin{aligned}
\gamma_{1,1} &= (\cos\alpha)(1 - \frac{1}{4}(i\mu_k)^{-1}g(0) + \frac{1}{32}(i\mu_k)^{-2}J(0) + o(\mu_k^{-2}))\\
&\quad + (\sin\alpha)(i\mu_k + \frac{1}{4}g(0) + \frac{1}{64}(i\mu_k)^{-1}(2J(0) - 8g^2(0) + 8I'(0) - 16g'(0)) + o(\mu_k^{-1}))\\
\gamma_{2,2} &= (\cos\beta)((1 + (-i\mu_k)^{-1}\frac{1}{8}(I(a) - 2g(a)) + \frac{1}{64}(-i\mu_k)^{-2}(K(a) + 2J(a)))e^{\frac{1}{2}G(a)}\\
&\quad + o(\mu_k^{-2})) + (\sin\beta)((-i\mu_k + \frac{1}{4}g(a) + \frac{1}{8}I(a)\\
&\quad + \frac{1}{64}(-i\mu_k)^{-1}(2J(a) + K(a) + 4g(a)I(a) - 8g^2(a) + 8I'(a) - 16g'(a)))e^{\frac{1}{2}G(a)}\\
&\quad + o((i\mu_k)^{-1}))\\
\gamma_{1,2} &= (\cos\alpha)(1 + \frac{1}{4}(-i\mu_k)^{-1}g(0) + \frac{1}{32}(-i\mu_k)^{-2}J(0) + o(\mu_k^{-2}))
\end{aligned}$$

$$+ (\sin\alpha)(-i\mu_k + \frac{1}{4}g(0) + \frac{1}{64}(-i\mu_k)^{-1}(2J(0) - 8g^2(0) + 8I'(0)$$
$$- 16g'(0)) + o(\mu_k^{-1}))$$
$$\gamma_{2,1} = (\cos\beta)((1 + \frac{1}{8}(i\mu_k)^{-1}(I(a) - 2g(a)) + \frac{1}{64}(i\mu_k)^{-2}(K(a) + 2J(a)))e^{\frac{1}{2}G(a)}$$
$$+ o(\mu_k^{-2})) + (\sin\beta)((i\mu_k + \frac{1}{4}g(a) + \frac{1}{8}I(a)$$
$$+ \frac{1}{64}(i\mu_k)^{-1}(2J(a) + K(a) + 4g(a)I(a) - 8g^2(a) + 8I'(a) - 16g'(a)))e^{\frac{1}{2}G(a)}$$
$$+ o(\mu_k^{-1})).$$

Finally we substitute $\gamma_{1,1}$, $\gamma_{2,2}$, $\gamma_{1,2}$ and $\gamma_{2,1}$ in equations (4.3.42) and (4.3.38). We compare the coefficients of k^0, k^{-1} and k^{-2} to get the values of ρ_0, ρ_1 and ρ_2, hence the eigenvalue asymptotic.

We will use a Computer Algebra (Sage) to generate a code which allows automatic generation of the eigenvalue asymptotic expansions. We follow the following algorithm. We divide ψ_1 and ψ_2 by the highest μ-power. Then apply a second order Taylor expansion thereafter. We then extract the coefficients of k^{-1} and k^{-2} and solve for ρ_1 and ρ_2 respectively. From Theorem 3.1.1 we observe that $\rho_0 = 0$. The results are populated in the following theorem.

Theorem 4.3.1. *For $g, h \in C^1[0,a]$, there exists a positive integer k_0 such that the eigenvalues of the problem* (1.3.1)-(1.3.2), *where $B_1 y = \cos\alpha y(0) + \sin\alpha y'(0)$ and $B_2 y = \cos\beta y(a) + \sin\beta y'(a)$ can be enumerated in such a way that $\hat{\lambda}_k = \hat{\mu}_k^2$, for $k \geq k_0$, where the $\hat{\mu}_k$ satisfy the asymptotics*

$$\hat{\mu}_k = \frac{\pi}{a}k + \rho_0 + \frac{\rho_1}{k} + \frac{\rho_2}{k^2} + o(k^{-2}),$$

where the numbers ρ_0, ρ_1 and ρ_2 are given by

$$\rho_0 = 0,$$
$$\rho_1 = \frac{1}{8}\frac{I(a)}{\pi} - \frac{1}{8}\frac{I(0)}{\pi} + \frac{1}{4}\frac{g(0)}{\pi} - \frac{1}{4}\frac{g(a)}{\pi} + \frac{1}{\pi}\frac{\cos\alpha}{\sin\alpha} + \frac{1}{\pi}\frac{\cos\beta}{\sin\beta}$$
$$= \frac{1}{8}\frac{I(a)}{\pi} + \frac{1}{4}\frac{g(0)}{\pi} - \frac{1}{4}\frac{g(a)}{\pi} + \frac{1}{\pi}\frac{\cos\alpha}{\sin\alpha} + \frac{1}{\pi}\frac{\cos\beta}{\sin\beta},$$

$$\rho_2 = 0.$$

The above theorem concludes this subsection. We have now computed the terms of the eigenvalue asymptotic expansions for the boundary conditions considered. The terms are depended on the functions g and h which was somehow expected since in this case we considered the differential equation under study with $g, h \neq 0$.

4.3.3 Asymptotics of the eigenvalues for $B_1 y = y(0,\mu) - y(a,\mu)$ and $B_2 y = y'(0,\mu) - y'(a,\mu)$

In this section we compute the asymptotic eigenvalues for the second set of the boundary conditions. Most of the calculations relevant for the computation of eigenvalue asymptotic terms for this case were performed in the section above. From equations (4.3.34) and (4.3.35) we deduce the equivalence of equation (4.3.37) for this case as

$$\begin{aligned}
B_{1,k} &= \delta_{k,0}(0,\mu) - \delta_{k,0}(a,\mu)e^{\omega_k i\mu a} \\
B_{2,k} &= \delta_{k,1}(0,\mu) - \delta_{k,1}(a,\mu)e^{\omega_k i\mu a}
\end{aligned} \tag{4.3.44}$$

where $B_{k,1}, B_{k,2}, k = 1, 2$ are given by

$$\begin{aligned}
B_{1,1} &= 1 - \frac{1}{4}(i\mu_k)^{-1}g(0) + \frac{1}{32}(i\mu_k)^{-2}J(0) + o(\mu_k^{-2}) \\
&\quad - (1 + \frac{1}{8}(i\mu_k)^{-1}(I(a) - 2g(a)) + \frac{1}{64}(i\mu_k)^{-2}(K(a) + 2J(a)))e^{\frac{1}{2}G(a)}e^{i\mu_k} + o(\mu_k^{-2}), \\
B_{1,2} &= 1 - \frac{1}{4}(-i\mu_k)^{-1}g(0) + \frac{1}{32}(-i\mu_k)^{-2}J(0) + o(\mu_k^{-2}) \\
&\quad - (1 + \frac{1}{8}(-i\mu_k)^{-1}(I(a) - 2g(a)) + \frac{1}{64}(-i\mu_k)^{-2}(K(a) + 2J(a)))e^{\frac{1}{2}G(a)}e^{-i\mu_k} + o(\mu_k^{-2}), \\
B_{2,1} &= i\mu_k + \frac{1}{4}g(0) + \frac{1}{64}(i\mu_k)^{-1}(2J(0) - 8g^2(0) + 8I'(0) - 16g'(0)) + o(\mu_k^{-1}) \\
&\quad - (i\mu_k + \frac{1}{4}g(a) + \frac{1}{8}I(a) + \frac{1}{64}(i\mu_k)^{-1}(2J(a) + K(a) + 4g(a)I(a) - 8g^2(a) + 8I'(a) \\
&\quad - 16g'(a)))e^{\frac{1}{2}G(a)}e^{i\mu_k} + o(\mu_k^{-1}),
\end{aligned}$$

$$B_{2,2} = -i\mu_k + \frac{1}{4}g(0) + \frac{1}{64}(-i\mu_k)^{-1}(2J(0) - 8g^2(0) + 8I'(0) - 16g'(0)) + o(\mu_k^{-1})$$
$$- (-i\mu_k + \frac{1}{4}g(a) + \frac{1}{8}I(a) + \frac{1}{64}(-i\mu_k)^{-1}(2J(a) + K(a) + 4g(a)I(a) - 8g^2(a)$$
$$+ 8I'(a) - 16g'(a)))e^{\frac{1}{2}G(a)}e^{-i\mu_k} + o(\mu_k^{-1}).$$

We then compute the characteristic determinant, D of the matrix M (equivalent to the matrix in equation (3.0.1)) as

$$\begin{aligned} D(\mu_k) &= B_1y_1B_2y_2 - B_1y_2B_2y_1 \\ &:= \xi_1 - \xi_2. \end{aligned} \tag{4.3.45}$$

Since the zeros of ϕ_0 are double zero, we have the eigenvalue asymptotic of the form

$$\mu_k = \frac{\pi}{2a}k + \rho_{0,\pm} + \frac{\rho_{1,\pm}}{k^{\frac{1}{2}}} + \frac{\rho_{2,\pm}}{k}.$$

Thus here we have

$$\begin{aligned} e^{i\mu_k a} &= e^{i\frac{\pi}{2}k}e^{ia\rho_0}\exp(ia(\frac{\rho_1}{k^{\frac{1}{2}}} + \frac{\rho_2}{k})) \\ &= (i)^k e^{ia\rho_0}(1 + ia\frac{\rho_1}{k^{\frac{1}{2}}} + (ia\rho_2 - \frac{a^2\rho_1^2}{2})\frac{1}{k}). \end{aligned}$$

and

$$\begin{aligned} e^{-i\mu_k a} &= e^{-i\frac{\pi}{2}k}e^{-ia\rho_0}\exp(-ia(\frac{\rho_1}{k^{\frac{1}{2}}} + \frac{\rho_2}{k})) \\ &= (-i)^k e^{-ia\rho_0}(1 - ia\frac{\rho_1}{k^{\frac{1}{2}}} - (ia\rho_2 + \frac{a^2\rho_1^2}{2})\frac{1}{k}). \end{aligned}$$

As in the above section, we deduce the terms ρ_0, ρ_1 and ρ_2 by comparing the coefficients of k^0, $k^{-\frac{1}{2}}$ and k^{-1} in equation (4.3.45). Since it is difficult to compute the above terms analytically, we also make use of the Computer Algebra to calculate the eigenvalue asymptotic expansion. We follow the same algorithm as in the above section.

Since the zeros in this problem are double zeros, we have a quadratic equation when extracting the coefficient of k^0 using Sage and equating it to zero. Human intervention is needed to solve for ρ_0 since it appears in both sides of the equation when using Sage. The extracted coefficient of k^0 is

$$-((-2ie^{G(0)} - 2ie^{G(a)})e^{ia\rho_0} + 2ie^{(2ia\rho_0 + \frac{1}{2}G(0) + \frac{1}{2}G(a))} + 2ie^{(\frac{1}{2}G(0) + \frac{1}{2}G(a))})e^{-ia\rho_0}. \quad (4.3.46)$$

By multiplying each term of equation (4.3.46) by $z = e^{ia\rho_0}$ and equating to zero we get

$$z^2 e^{\frac{1}{2}G(a)} - z(1 + e^{G(a)}) + e^{\frac{1}{2}G(a)} = 0,$$

hence a quadratic equation in z. Thus using the quadratic formula we get

$$z = \frac{1 + e^{G(a)} \pm (1 + e^{G(a)} - 2e^{\frac{1}{2}G(a)})^{\frac{1}{2}}(1 + e^{G(a)} + 2e^{\frac{1}{2}G(a)})^{\frac{1}{2}}}{2e^{\frac{1}{2}G(a)}}.$$

Therefore after solving for ρ_0 we get

$$\rho_{0,\pm} = \frac{2\ln(1 + e^{G(a)} \pm (1 + e^{G(a)} - 2e^{\frac{1}{2}G(a)})^{\frac{1}{2}}(1 + e^{G(a)} + 2e^{\frac{1}{2}G(a)})^{\frac{1}{2}}) - 2\ln 2 + G(a)}{2ia}.$$

We then substitute $\rho_{0,\pm}$ back into the coefficients of $k^{-\frac{1}{2}}$ and k^{-1} and solve for $\rho_{1,\pm}$ and $\rho_{2,\pm}$ respectively. The results are populated in the theorem below.

Theorem 4.3.2. *For $g, h \in C^1[0,a]$, there exists a positive integer k_0 such that the eigenvalues of the problem (1.3.1)-(1.3.2), where $B_1y = y(a) - y(0)$ and $B_2y = y'(a) - y'(0)$ can be enumerated in such a way that $\hat{\lambda}_k = \hat{\mu}_k^2$, for $k \geq k_0$, where the $\hat{\mu}_k$ satisfy the asymptotics*

$$\mu_k = \frac{\pi}{2a}k + \rho_{0,\pm} + \frac{\rho_{1,\pm}}{k^{\frac{1}{2}}} + \frac{\rho_{2,\pm}}{k} + o(k^{-2})$$

where the numbers $\rho_{0,\pm}$, $\rho_{1,\pm}$ and $\rho_{2,\pm}$ are given by

$$\rho_{0,+} = \frac{2\ln(1 + e^{G(a)} + (1 + e^{G(a)} - 2e^{\frac{1}{2}G(a)})^{\frac{1}{2}}(1 + e^{G(a)} + 2e^{\frac{1}{2}G(a)})^{\frac{1}{2}}) - 2\ln 2 + G(a)}{2ia},$$

$$\begin{aligned}
\rho_{1,+} &= 0,\\
\rho_{2,+} &= \frac{L(a)(1 - e^{2ia\rho_0})}{4(\pi - \pi e^{2ia\rho_0})}\\
&= \frac{1}{64}(16\pi + \pi G(a)^4 + 16\pi(\ln 2)^4 + 8(2\pi e^{G(a)} - \pi\ln 2)G(a)^3\\
&- 8(\pi + 12\pi\ln 2e^{G(a)} - 3\pi(\ln 2)^2 - 12\pi e^{2G(a)})G(a)^3\\
&- 32(\pi(\ln 2)^3 + 12\pi\ln 2e^{2G(a)} - 8\pi e^{3G(a)} + 2(\pi - 3\pi(\ln 2)^2)e^{G(a)} - \pi\ln 2)G(a)\\
&- 128(\pi - 3\pi(\ln 2)^2)e^{2G(a)} - 128(\pi(\ln 2)^3 - \pi\ln 2)e^{G(a)}\\
&- 512\pi\ln 2e^{3G(a)} - 32\pi(\ln 2)^2 + 256\pi e^{4G(a)})I(a),
\end{aligned}$$

or

$$\rho_{0,-} = \frac{2\ln(1 + e^{G(a)} - (1 + e^{G(a)} - 2e^{\frac{1}{2}G(a)})^{\frac{1}{2}}(1 + e^{G(a)} + 2e^{\frac{1}{2}G(a)})^{\frac{1}{2}}) - 2\ln 2 + G(a)}{2ia},$$

$$\begin{aligned}
\rho_{1,-} &= 0,\\
\rho_{2,-} &= \frac{1}{64}(144\pi + \pi G(a)^4 + 16\pi(\ln 2)^4 + 8(2\pi - \pi\ln 2)G(a)^3 - 128\pi(\ln 2)^3\\
&+ 8(11\pi + 3\pi(\ln 2)^2 - 12\pi\ln 2)G(a)^2 + 352\pi(\ln 2)^2 + 32(6\pi - \pi(\ln 2)^3\\
&+ 6\pi(\ln 2)^2 - 11\pi\ln 2)G(a) - 384\pi(\ln 2))I(a).
\end{aligned}$$

The above theorem concludes the sub-section. We have now computed the first four terms of the periodic boundary conditions.

Chapter 5

CONCLUSION

The dissertation studies the asymptotic distribution of the eigenvalues of a second order differential operators. We computed the estimates of the eigenvalues by considering the differential equation with $g = 0$ and $h = 0$. To improve on the asymptotics we then considered the differential equation with $g, h \neq 0$ and computed the asymptotic eigenvalues.
The study becomes complex when one studies higher order differential operators, that is for fourth and higher order differential operators.

5.1 Further Reading

For further reading and understanding of boundary eigenvalue problems and most of the work carried out in this dissertation one can consult the book [12] by Mennicken and Möller. There are several papers and books related to this study that were published for further reading and consultation. One can also refer to the work of Binding, Browne and Watson ([3], [4] and [5]) for further reading on topics similar to the one under study.

In [3], Binding et al. considered the regular Sturm-Liouville equation

$$-y'' + qy = \lambda y \quad \text{on} \quad [0, 1], \tag{5.1.1}$$

subject to the boundary conditions

$$y(0)\cos\alpha = y'(0)\sin\alpha, \quad \alpha \in [0,\pi), \tag{5.1.2}$$

and

$$(y'/y)(1) = f(\lambda) \tag{5.1.3}$$

for a class of functions f.

Binding et al. continued the work of [3] in [4] with $q \in L^1[0,1]$ and where $f(\lambda)$ is a rational function of the form

$$f(\lambda) = a\lambda + b - \sum_{k=1}^{N} \frac{b_k}{\lambda - c_k}. \tag{5.1.4}$$

There are other several publications that one can consult in relation to the work carried out in this dissertation.

Chapter 6

APPENDIX

In the appendix, we present the Sage code used to calculate the first four terms of the eigenvalue asymptotics given in Theorem 4.3.1 and Theorem 4.3.2.

6.1 Code used for calculation of ρ_1 and ρ_2 in Sub-section 4.3.2

The code for calculation of ρ_1 and ρ_2 for the separated boundary conditions $B_1 y = \cos\alpha y(0,\mu) + \sin\alpha y'(0,\mu)$ and $B_2 y = \cos\beta y(a,\mu) + \sin\beta y'(a,\mu)$ is given below.

```
#Definition of variables used in the calculations

var('x,mu,alpha,beta,p,p0,p1,p2,a');

#Definition of functions g,h,G,I,J and K

function('g',x);
function('h',x);
function('G',x);
function('I',x);
```

```
function('J',x);
function('K',x);

#Definition of delta[k][0] and delta[k][1]

delta10 = (1+1/8*((i*mu)^(-1))*(I(x)-2*g(x))
+1/64*((i*mu)^(-2))*(2*J(x)+K(x)))*e^(1/2*G(x))
delta20 = (1+1/8*((-i*mu)^(-1))*(I(x)-2*g(x))
+1/64*((-i*mu)^(-2))*(2*J(x)+K(x)))*e^(1/2*G(x))
delta11 = (i*mu-1/4*g(x)+1/8*I(x)+1/64*((i*mu)^(-1))*(2*J(x)+K(x)
+4*g(x)*I(x)-32*h(x)-16*diff(g(x))))*e^(1/2*G(x))
delta21 = (-i*mu-1/4*g(x)+1/8*I(x)+1/64*((-i*mu)^(-1))*(2*J(x)+K(x)
+4*g(x)*I(x)-32*h(x)-16*diff(g(x))))*e^(1/2*G(x))

#Definition of the boundary conditions
cos(alpha)*y(0)+sin(alpha)*y'(0)

gamma11 = ((cos(alpha))*delta10).substitute(x=0)
+((sin(alpha))*delta11).substitute(x=0)
gamma12 = ((cos(alpha))*delta20).substitute(x=0)
+((sin(alpha))*delta21).substitute(x=0)

#Definition of the boundary condition
cos(beta)*y(a)+sin(beta)*y'(a)

gamma21 = ((cos(beta))*delta10).substitute(x=a)
+((sin(beta))*delta11).substitute(x=a)
gamma22 = ((cos(beta))*delta20).substitute(x=a)
+((sin(beta))*delta21).substitute(x=a)

#Taylor expansions of e^{i*mu_k*a}, e^{-i*mu_k*a} and 1/mu_k
#With p1 = rho1 and p2 = rho2.

rho0 = p0 = 0
```

```
f1 = taylor(exp(i*p0*a)*exp(i*a*(p1*p+p2*p^2)),p,0,2)
f2 = taylor(exp(-i*p0*a)*exp(-i*a*(p1*p+p2*p^2)),p,0,2)
f3 = taylor((1+a*(p0+p1*p)*p/pi)^(-1),p,0,1)
invmu = f3*a*p/pi

#Computation of psi1. The syntax expands and simplifies Psi1.
#Thereafter, it expands the simplified equation again since
#Sage cannot expand and simplify in one line.

psi1 = (gamma11*gamma22).expand().simplify_full().expand()
psi2 = (gamma21*gamma12).expand().simplify_full().expand()

#Finding the highest mu-power in psi1 and psi2
#That is, finding the degree of psi1 and psi2 in mu.

deg1 = psi1.degree(mu)

#Here we divide psi1 and psi2 by the highest mu-power
#to reduce the order of the equations in mu.

psidiv1 = psi1/mu^deg1
psidiv2 = psi2/mu^deg1

#Substituting Taylor expanded 1/invmu for mu
#in simplified psi1 and psi2 from above.

psi1subs1 = psidiv1.substitute(mu=1/invmu)
psi1subs2 = psidiv2.substitute(mu=1/invmu)

#Now we simplify and expand the above equations
#and use the syntax like above for psi1 and psi2.

psi11 = psi1subs1.expand().simplify_full().expand()
psi22 = psi1subs2.expand().simplify_full().expand()
```

```
#We now do the second order Taylor expansions of the above
#equations to reduce the equations to second degrees, that is
#highest p-power of 2.

taypsi11 = taylor(psi11,p,0,2)
taypsi22 = taylor(psi22,p,0,2)

#Computation of psi1*e^(-i*mu_k*a) and psi2*e^(i*mu_k*a)
#Thereafter, we expand and simplify like above.

D1 = (taypsi11*f2).expand().simplify_full().expand()
D2 = (taypsi22*f1).expand().simplify_full().expand()

#Computation of the characteristic function.
#Again we do the second order Taylor expansion
#of the characteristic function.

D11 = (D1-D2).simplify_full().expand()
tayD11 = taylor(D11,p,0,2)

#Computations of rho1 and rho2 by comparing the coefficients
#of p and p^2. We then get rho1 and rho2 by
#equating the extracted coefficients of p and p^2 to zero
#and respectively solving for p1 and p2.

coef1 = tayD11.coefficient(p,1)
rho1 = expand(solve(coef1,p1)[0].right())
coef2 = tayD11.coefficient(p,2)
subcoef1 = coef2.substitute(p1=rho1)
rho2 = expand(solve(subcoef1,p2)[0].right())
```

6.2 Code used for calculation of $\rho_{0,\pm}$, $\rho_{1,\pm}$ and $\rho_{2,\pm}$ in Sub-section 4.3.3

The code for calculation of ρ_1 and ρ_2 for the periodic boundary conditions $B_1y = y(0,\mu) - y(a,\mu)$ and $B_2y = y'(0,\mu) - y'(a,\mu)$ is given below

```
# Definition of variables used in the calculations
var('x,mu,ph,p0,p1,p2,a');

#Definition of functions g,h,G,L,J and K
function('g',x);
function('h',x);
function('G',x);
function('L',x); #Here L stands for function I
function('J',x);
function('K',x);

#Definition of delta[k][0] and delta[k][1]
delta10 = (1+1/8*((I*mu)^(-1))*(L(x)+2*g(x))
+1/64*((I*mu)^(-2))*(2*J(x)+K(x)))*e^(1/2*G(x))
delta20 = (1+1/8*((-I*mu)^(-1))*(L(x)+2*g(x))
+1/64*((-I*mu)^(-2))*(2*J(x)+K(x)))*e^(1/2*G(x))
delta11 = (I*mu-1/4*g(x)+1/8*L(x)
+1/64*((I*mu)^(-1))*(2*J(x)+K(x)-4*g(x)*L(x)-4*h(x)
+16*diff(g(x))))*e^(1/2*G(x))
delta21 = (-I*mu-1/4*g(x)+1/8*L(x)
+1/64*((-I*mu)^(-1))*(2*J(x)+K(x)-4*g(x)*L(x)-4*h(x)
+16*diff(g(x))))*e^(1/2*G(x))

#Taylor expansion of e^{I*mu_k*a} and e^{-I*mu_k*a}
with p0 = rho0, p1 = rho1 and p2 = rho2
#Here ph =sqrt(p) (see p in the Separate boundary conditions
code above)
```

```
f1p = taylor(exp(I*p0*a)*exp(I*a*(p1*ph+p2*ph^2)),ph,0,2)
f2p = taylor(exp(-I*p0*a)*exp(-I*a*(p1*ph+p2*ph^2)),ph,0,2)

#Definition of the boundary conditions y(0) - y(a) = 0

B11 = delta10.substitute(x=0) - delta10.substitute(x=a)*f1p
B12 = delta20.substitute(x=0) - delta20.substitute(x=a)*f2p

#Definition of the boundary conditions y'(0) - y'(a) = 0

B21 = delta11.substitute(x=0) - delta11.substitute(x=a)*f1p
B22 = delta21.substitute(x=0) - delta21.substitute(x=a)*f2p


#Computation of xi1 and xi2.

xi1 = (B21*B12).expand().simplify_full().expand()
xi2 = (B11*B22).expand().simplify_full().expand()

#Computing the degree of xi1 and xi2, the highest mu-power.

deg1p = xi2.degree(mu)

#Dividing xi1 and xi2 by the degree of mu to reduce the order
of the equations in mu

xidiv1 = xi1/mu^deg1p
xidiv2 = xi2/mu^deg1p

#Computing Taylor expansion of 1/mu_k

f3p = taylor((1+a*(p0+p1*ph)*ph^2/2/pi)^(-1),ph,0,1)
invmup = f3p*a*2*ph^2/pi
```

```
#After expansion of 1/mu_k, we substitute mu_k
in simplified xi1 and xi2

xisubs1 = xidiv1.substitute(mu=1/invmup)
xisubs2 = xidiv2.substitute(mu=1/invmup)

#Expand and simplify xi1 and xi2 to read better

xi11 = xisubs1.expand().simplify_full().expand()
xi22 = xisubs2.expand().simplify_full().expand()

#Take second order Taylor expansion in ph to omit higher order terms

exp11 = taylor(xi11,ph,0,2)
exp22 = taylor(xi22,ph,0,2)

#Then expand and simplify again (Note: We always expand and
simplify for equations to read better

xi_1 = exp11.expand().simplify_full().expand()
xi_2 = exp22.expand().simplify_full().expand()

#Computing the characteristic determinant

D = (xi_1 - xi_2).simplify_full().expand()
Dexp = taylor(D,ph,0,2)

#Here we compute rho0, rho1 and rho2 by comparing the
coefficients of ph^0, ph^1 and ph^2
#We then equate the coefficients to zero and
solve for p0, p1 and p2
```

```
coef0 = Dexp.coefficient(ph,0)

#rho0 after manually solving for it from coef0

rho0 = (ln(1+exp(G(a))
+(1+exp(G(a))-2*exp(1/2*G(a)))^(1/2)
*(1+exp(G(a))-2*exp(1/2*G(a)))^(1/2))-ln(2)+1/2*G(a))/i*a

#The syntax to calculate for rho0
#expand(solve(coef0,p0)[0].right())

coef1 = Dexp.coefficient(ph,1)
temp = coef1.substitute(p0=rho0)
rho1 = expand(solve(coef1,p1)[0].right())
coef2 = Dexp.coefficient(ph,2)
temp1 = coef2.substitute(p1=0)
rho2 = expand(solve(coef2,p2)[0].right())
```

As indicated for the periodic boundary conditions, a human intervention is needed to solve for $\rho_{0,\pm}$ and $\rho_{2,\pm}$.
Below is a small code used to calculate for $\rho_{2,\pm}$ after some intervention. $\rho_{0,\pm}$ is calculated manually in Sub-section 4.3.3.

```
var('a');
function('G',x);
function('L',x); #L stands for function I

#Defining rho0 as calculated manually in Sub-section 4.3.3
for different signs.

p0 = ((ln(1+exp(G(a))+sqrt(1+exp(G(a))-2*exp(G(a)/2))
     *sqrt(1+exp(G(a))+2*exp(G(a)/2))-ln(2)+G(a)/2)))/(I*a)
```

```
p0 = ((ln(1+exp(G(a))-sqrt(1+exp(G(a))-2*exp(G(a)/2))
     *sqrt(1+exp(G(a))+2*exp(G(a)/2))-ln(2)+G(a)/2)))/(I*a)


#Defining rho2 that is calculated in the code above
#(Note that it is in terms of p0, hence a need to substitute
and simplify)

p2 = L(a)*(1-exp(2*I*a*p0))/4*(pi-pi*exp(2*I*a*p0))

#Simplifying for rho2 to get rho2 (i.e. not in terms of p0)

p2s = p2.expand().simplify_full()
```

The above codes conclude the appendix section. A reader who is interested in the results can plug the code step by step in Sage to get the numbers ρ_0, ρ_1 and ρ_2 for the separated and $\rho_{0,\pm}$, $\rho_{1,\pm}$ and $\rho_{2,\pm}$ for the periodic boundary conditions. The package Sage is also available online for convenience.

Bibliography

[1] Adams, R. A and Fournier, J. J. F *Sobolev Spaces*. Elsvier, 2003.

[2] Akulenko, L. D. and Nesterov, S. V. *High-Precision Methods in Eigenvalue Problems and their Applications*. Differential and Integral Equations and their Applicatiions, vol. 6. Chapman & Hall/CRC, 2005.

[3] Binding, P. A., Browne, P. J. and Watson, B. A. *Sturm-Liouville Problems with Boundary Conditions Rationally Dependent on the Eigenparameter. I.* Proceedings of the Edinburgh Mathematical Society, 45, 631-645, 2002.

[4] Binding, P. A., Browne, P. J. and Watson, B. A. *Sturm-Liouville Problems with Boundary Conditions Rationally Dependent on the Eigenparameter. II.* Journal of Computational and Applied Mathematics, 148, 147-168, 2002.

[5] Binding, P. A., Browne, P. J. and Watson, B. A. *Spectral Asymptotics for Sturm-Liouville Equations with Indefinite Weight*. Transaction of the American Mathematical Society, Vol. 354, Number 10, 4043-4065, 2002.

[6] Birman, M. Sh. and Solomyak, M. Z. *Asymptotic Behavior of the Spectrum Differential Equations*. Journal of Mathematical Sciences, vol. 12, no. 3, 247-283, 1979.

[7] Davies, E. B. *Spectral Theory and Differential Operators*. Cambridge Studies in Advanced Mathematics 42. Cambridge University Press, 1995.

[8] Faierman, A. *Distribution Of Eigenvalues Of A Two-Parameter System Of Differential Equations*. Transactions of the American Mathematical Society, vol 247, 45-86, 1979.

[9] Haroske, D. D. and Triebel, H. *Distributions, Sobolev Spaces, Elliptic Equations*. European Mathematical Society, 2008.

[10] Il'in, V. A. *Spectral Theory of Differential Operators*. Self-Adjoint Differential Operators. Consultants Bureau, New York, 1995.

[11] Kostyuchenko, A. G. and Levitan, B. M. *Asypmptotic Behavior of the Eigenvalues of the Sturm-Liouville Operator Problem*. Functional Analysis and its Applications, vol. 1, no. 1, 86-96, 1967.

[12] Mennicken, R. and Möller, M. *Non-Self Adjoint Boundary Eigenvalue Problems*. North-Holland Mathematical Studies 192. Amsterdam: Elsevier, 2003.

[13] Möller, M. and Zinsou, B. *Spectral Asymptotics of self-Adjoint Fourth Order Differential Operators With Eigenvalue Parameter Dependent Boundary Conditions*. Complex Anal. Oper. Theory. 6, 799-818, 2012.

[14] Naimark, M. A. *Linear Differential Operators, Part I*. Frederick Ungar Publishing Co., New York, 1967.

[15] Naimark, M. A. *Linear Differential Operators, Part II*. Frederick Ungar Publishing Co., New York, 1968.

[16] Shibata, T. *Variational Eigencurve And Bifurcation For Two-Parameter Nonlinear Sturm-Liouville Equations*. Topological Methods in Nonlinear Analysis, vol. 8, 79-93, 1966.

[17] Shkalikov, A. A. *Boundary-Value Problems for Ordinary Differential Equations with a Parameter in the Boundary Conditions*. Functional Analysis and its Applications, vol. 16, no. 4, 324-326, 1982.

[18] Weidmann, J. *Linear Operators in Hilbert Spaces*. Springer-Verlag New York Heidelberg Berlin, 1980.

[19] Yakubov, S. and Yakubov, Y. *Differential-Operator Equations, Ordinary and Partial Differential Equations*. Monographs and Surveys in Pure and Applied Mathematics 103. Chapman & Hall/CRC, 2000.

[20] Ziemer, W. P. *Weakly Differentiable Functions*. Springer-Verlag New York, 1989.

[21] Zinsou, B. *Self-Adjoint Fourth Order Differential Operators With Eigenvalue Parameter Dependent Boundary Conditions*. MSc Dissertation, University of the Witwatersrand, Johannesburg, 2009.